A COLLECTION OF EXAMPLES OF THE APPLICATIONS OF THE CALCULUS OF FINITE DIFFERENCES.

BY

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PREFACE.

In applying the general principles of the Calculus of Differences, laid down in the Appendix to the Translation of Lacroix, we have assigned the first place to Examples purely analytical of the methods themselves, endeavouring always to select such as may not only be useful to the student as exercises, but also as results, which he may have occasion to refer to in his future enquiries, and will thus be valuable in themselves, as materials which he will find it advantageous in a more advanced state of his knowledge to have ready at hand. Many of these results are theorems of some generality which we believe will not be found elsewhere, or at least, are not in common use, and where these occur, the leading steps of the demonstration are either set down, or the principle on which it depends mentioned, and in both cases the student will find a useful exercise for his invention in supplying what is omitted. We have then proceeded to questions of a more mixed nature, illustrative of the application of the Calculus of Differences, to a variety of subjects in which it may be employed with advantage as an instrument of investigation, such as the determination of the general terms of the series when the law of their formation is given, the theory of circulating functions, continued fractions, the determination of curves from such properties as involve a series of consecutive points separated by finite intervals, the doctrine of Interest and Annuities, and such other subjects as can be properly treated within our limits.

The want of a regular treatise, on the Calculus of Differences in our language, has long been a serious obstacle to the progress of the enquiring student. The Appendix annexed to the translation of Lacroix's Differential and Integral Calculus, although from the necessity of studying compression it is not so complete as its author could have wished, will, it is hoped, remove this obstacle in some degree, and at least put the analytical principles of the pure theoretical part of the calculus in the reader's power. But the method of applying those principles, and the formulæ derived from them, to the various cases and questions of pure, as well as mixed mathematics, in which they may be advantageously introduced, also demands some degree of explanation, and accordingly, in such of our examples as have this for their object, the reader will find the successive steps of the processes fully detailed, till the questions are reduced to such analytical difficulties, as the Appendix, or the preceding problems will enable him to surmount by himself.

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PART III.

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SECTION I.

EXAMPLES OF THE DIRECT METHOD OF DIFFERENCES.

In the following questions, x is supposed to be the independent variable, so that Δx is unity throughout.

(1). Required the differences of $\sin x\theta$ and $\cos x\theta$, θ being constant,

$$\Delta \sin x\theta = 2 . \sin\frac{\theta}{2} . \cos\left(x\theta + \frac{\theta}{2}\right)$$

$$\Delta \cos x\theta = -2 . \sin\frac{\theta}{2} \sin\left(x\theta + \frac{\theta}{2}\right).$$

(2). Required the differences of $\sin(h + x\theta)$ and $\cos(h+x\theta)$

$$\Delta \sin(h+x\theta) = 2 . \sin\frac{\theta}{2} . \cos\left\{h + \left(x + \frac{1}{2}\right)\theta\right\}$$

$$\Delta \cos(h+x\theta) = -2 . \sin\frac{\theta}{2} . \sin\left\{h + \left(x + \frac{1}{2}\right)\theta\right\}.$$

(3). To find the $(2n)^{th}$ and $(2n-1)^{th}$ differences of the same functions,

$$\Delta^{2n} \sin(h + x\theta) = \left(2\sin\frac{\theta}{2}\right)^{2n} . \sin\{h + (x+n)\theta\}$$

$$\Delta^{2n} \cos(h + x\theta) = \left(2\sin\frac{\theta}{2}\right)^{2n} . \cos\{h + (x+n)\theta\}$$

$\Delta^{2n-1} \sin(h + x\theta) =$

$$= \left(2 \sin \frac{\theta}{2}\right)^{2n-1} \cdot \cos \left\{ h + \left(x + \frac{2n-1}{2} \theta\right)\right\}$$

$\Delta^{2n-1} \cos (h + x\theta) =$

$$= -\left(2 \sin \frac{\theta}{2}\right)^{2n-1} \cdot \sin \left\{ h + \left(x + \frac{2n-1}{2} \theta\right)\right\}.$$

In like manner we may obtain the following differences:

(4). $\Delta \tan x\theta = \dfrac{\sin \theta}{\cos x\theta \,.\, \cos (x+1)\theta}.$

(5). $\Delta \operatorname{cotan} x\theta = \dfrac{-\sin \theta}{\sin x\theta \,.\, \sin (x+1)\theta}.$

(6). $\Delta \dfrac{a^{2^x}+1}{a^{2^x}-1} = -\dfrac{2a^{2^x}}{a^{2^{x+1}}-1}.$

(7). $\Delta \dfrac{2^x (a^{2^x}+1)}{a^{2^x}-1} = 2^x \cdot \dfrac{a^{2^x}-1}{a^{2^x}+1}.$

(8). $\Delta \,.\, 2^x \sin \dfrac{\theta}{2^x} = 2^{x+2} \,.\, \sin \dfrac{\theta}{2^{x+1}} \cdot \left(\sin \dfrac{\theta}{2^{x+2}}\right)^2.$

(9). $\Delta \,.\, 2^{2x} \left(\sin \dfrac{\theta}{2^x}\right)^2 = 2^{2x+2} \,.\, \left(\sin \dfrac{\theta}{2^{x+1}}\right)^4.$

(10). $\Delta \tan \dfrac{\theta}{2^x} = -\dfrac{\tan \dfrac{\theta}{2^{x+1}}}{\cos \dfrac{\theta}{2^x}},$

(11). $\Delta \,.\, \tan \dfrac{\theta}{2^x} = -2^x \,.\, \tan \dfrac{\theta}{2^x} \cdot \left(\tan \dfrac{\theta}{2^{x+1}}\right)^2.$

(12). $\Delta \,.\, (-2)^x \,.\, \sin \dfrac{\theta}{2^x}$

$$= (-1)^{x+1} \,.\, 2^{x+2} \,.\, \sin \frac{\theta}{2^{x+2}} \left(\cos \frac{\theta}{2^{x+2}}\right)^2.$$

(13). $$\Delta \frac{1}{2^x . \sin \frac{\theta}{2^x}} = -2 . \frac{\sin \frac{\theta}{2^{x+2}}}{2^x . \sin \frac{\theta}{2^x}}.$$

(14). $$\Delta \frac{1}{2^x . \tan \frac{\theta}{2^x}} = \frac{\tan \frac{\theta}{2^{x+1}}}{2^{x+1}}.$$

(15). $$\Delta \left\{ \frac{1}{2^x . \tan \frac{\theta}{2^x}} \right\}^2 = \frac{1}{2^{x+1}} - \left\{ \frac{\tan \frac{\theta}{2^{x+1}}}{2^{x+1}} \right\}^2.$$

(16). $$\Delta \frac{\cos \frac{\theta}{2^x}}{\left\{ 2^x . \sin \frac{\theta}{2^x} \right\}^2} = \frac{\sin \frac{\theta}{2^x+1}}{\left\{ 2^{x+1} . \cos \frac{\theta}{2^{x+1}} \right\}^2}.$$

(17.) $$\Delta \cot 2^x \theta = - \frac{1}{\sin 2^{x+1} \theta}.$$

(18.) $$\Delta \frac{\log (2 \sin 2^x \theta)}{2^x} = - \frac{\log \tan 2^x \theta}{2^{x+1}}.$$

(19.) $$\Delta \frac{1}{\left(2^x . \sin \frac{\theta}{2^x}\right)^2} = - \frac{1}{\left(2^{x+1} . \cos \frac{\theta}{2^{x+1}}\right)^2}.$$

(20.) Let $s_x = \sin (a + x\,\theta)$; $c_x = \cos (a + x\,\theta)$, then

$$\Delta \frac{1}{c_x . c_{x+1} \ldots . c_{x+n+1}}$$

$$= 2 \sin (n+1)\,\theta . \frac{s_{x+n+1}}{c_x . c_{x+1} \ldots . c_{x+n+2}}.$$

(21.) The same notation being employed,

$$\Delta \frac{(-1)^x}{s_x . s_{x+1}} = 2 . \cos \theta . \frac{(-1)^{x+1}}{s_x . s_{x+2}}$$

(22). $\Delta \dfrac{(-1)^x}{s_x \cdot s_{x+1} \cdots s_{x+2n+1}}$

$$= 2 \,.\, \cos(n+1)\theta \,.\, \frac{(-1)^{x+1} \,.\, s_{x+n+1}}{s_x \cdot s_{x+1} \cdots s_{x+2n+2}}.$$

(23). $\Delta \dfrac{(-1)^x}{c_x \cdot c_{x+1}} = 2 \,.\, \cos\theta \,.\, \dfrac{(-1)^{x+1}}{c_x \,.\, c_{x+2}}.$

(24). $\Delta \dfrac{(-1)^x}{c_x \,.\, c_{x+1} \cdots c_{x+2n+1}}$

$$= 2 \,.\, \cos(n+1)\theta \,.\, \frac{(-1)^{x+1} \,.\, c_{x+2n+2}}{c_x \cdot c_{x+1} \cdots c_{x+n+1}}.$$

(25). Required the difference of arc tan $x\,\theta$, or (as we shall in future designate this function) $\tan^{-1} x\,\theta$.

$$\Delta \tan^{-1} x\,\theta = \tan^{-1} \left\{ \frac{\theta}{1 + \theta^2 x + \theta^2 x^2} \right\}.$$

The expression $\tan^{-1} x\,\theta$ must not be confounded with $(\tan x\,\theta)^{-1}$ or $\dfrac{1}{\tan x\,\theta}$. The reasons for employing this mode of expression instead of the geometrical circumlocution arc $(\tan = x\,\theta)$ or arc tan $x\,\theta$ are these.

We have already in the differential calculus as well as in that of differences, experienced the great advantage not only in point of brevity, but of clearness and symmetry which arises from denoting the repetition of the operations expressed by d and Δ, by annexing the number of repetitions as an exponent to the characteristic, and we have already seen (Appendix, Art 378.) that the inverse operation of integration in the two calculi is rightly represented on this principle by the same characteristics d and Δ with negative exponents. The same notation may be used to denote the repetition of any operation, whether it be such as modifies the *form* of a function, which is the case with those just noticed or such as expresses the nature of the function itself, as log, cos, tan,

&c. We may use $\log^2 x$, $\cos^3 x$, $\tan^n x$, &c. for log log x, cos cos cos x, tan tan....tan x, &c. respectively, and in general

$$f(f(x)) \text{ or } ff(x) \text{ may be written } f^2(x)$$

$$f(f(f(x))) \text{ or } fff(x) \text{ may be written } f^3(x)$$

and so on, which gives in general $f^m f^n(x) = f^{m+n}(x)$.

If we now enquire the meaning of $f^0(x)$, we need only make $n=0$, $m=1$, which gives

$$ff^0(x) = f(x),$$

and consequently $f^0(x) = x$. If now we make $m=1$ and $n=-1$ we find

$$ff^{-1}(x) = f^0(x) = x,$$

so that $f^{-1}(x)$ must denote "that quantity whose function f is x," or rather, that function of x which operated on in the manner denoted by f shall produce simply x. $f^{-1}(x)$ then denotes the *inverse function* of $f(x)$: thus $\tan^{-1} x$ will stand for arc ($\tan = x$), $\sin^{-1} x$, $\cos^{-1} x$, $\log^{-1} x$ respectively for arc ($\sin = x$), arc ($\cos = x$), e^x, or number whose logarithm is x*. The symmetry of this notation and above all the new

* The notations $f^n(x)$, $f^{-n}(x)$, $\sin^{-1} x$ &c. were explained by the author of these latter pages in a paper, "On a remarkable application of Cotes's theorem" in the Philosophical Transactions, 1813, as he then supposed for the first time. The work of a German Analyst, Burmann, has, however, within these few months come to his knowledge, in which the same is explained at a considerably earlier date. He, however, does not seem to have noticed the convenience of applying this idea to the inverse functions $\tan^{-1}$, &c. nor does he appear at all aware of the *inverse calculus of functions* to which it give rise. Burmann is a zealous partizan of the combinatory analysis of Hindenburg, the very principle of which is the exclusion of all *analytical artifice*, which is no where so strongly called for as in the calculus here alluded to.

and most extensive views it opens of the nature of analytical operations seem to authorize its universal adoption, not to mention the real inconvenience which more than one author of eminence has been put to for want of some notation founded on principle to express any inverse function without introducing a new characteristic.

The equation which gave rise to this digression is easily proved if we call to mind that

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B};$$

for, if we take the inverse function $\tan^{-1}$ on both sides, we have

$$A - B = \tan^{-1}\left\{\frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}\right\}$$

and for A and B writing $\tan^{-1} A$ and $\tan^{-1} B$,

$$\tan^{-1} A - \tan^{-1} B = \tan^{-1}\left\{\frac{A-B}{1+AB}\right\}$$

which the student is left to apply to the case in question.

(26). Required the difference of $\tan^{-1}(h + x\theta)$.

$$\Delta \tan^{-1}(h + x\theta) = \tan^{-1}\left\{\frac{\theta}{1+\theta(h+x\theta) + (h+x\theta)^2}\right\}.$$

(27.) Required the difference of $\tan^{-1}\left\{\frac{a+bx}{A+Bx}\right\}$.

$$\Delta \tan^{-1}\left\{\frac{a+bx}{A+Bx}\right\} =$$

$$= \tan^{-1}\left\{\frac{Ab - aB}{\left\{\begin{matrix} A^2 + AB \\ + a^2 + ab\end{matrix}\right\} + x\left\{\begin{matrix} 2AB + B^2 \\ + 2ab + b^2\end{matrix}\right\} + x^2(B^2 + b^2)}\right\}.$$

(28). Required the difference of $\tan^{-1} . u_x$

$$\Delta \tan^{-1} u_x = \tan^{-1} \frac{\Delta u_x}{1 + u_x u_{x+1}}.$$

(29). Required the difference of $2^x . \tan^{-1}\left(\frac{\theta}{2^x}\right)$

$$\Delta . 2^x \tan^{-1}\left(\frac{\theta}{2^x}\right) = 2^x . \tan^{-1}\left\{\frac{\theta^3}{4 . 2^{3x} + 3\,\theta^2 . 2^x}\right\}.$$

(30). It is required to demonstrate the truth of the two following theorems in which p represents $2 \sin \theta$,

$$\cos 2\,n\,\theta = 1 - n\,p . \sin \theta - \frac{n\,(n-1)}{1 . 2} p^2 . \cos 2\,\theta$$

$$+ \frac{n\,(n-1)\,(n-2)}{1 . 2 . 3} p^3 . \sin 3\,\theta + \&c.$$

$$\sin 2\,n\,\theta = \frac{n}{1} p . \cos \theta - \frac{n\,(n-1)}{1 . 2} p^2 . \sin 2\,\theta$$

$$- \frac{n\,(n-1)\,(n-2)}{1 . 2 . 3} p^3 . \cos 3\,\theta + \&c.$$

the sign being alternately $+ + - -$ and the series breaking off at p^n whenever n is a positive integer.

These may be deduced from the general expression for u_{x+n} in Art. 345, by substituting $\cos 2\,x\,\theta$ or $\sin 2\,x\,\theta$ for u_x, developing their successive differences and finally making $x=0$.

(31). Required the successive differences of x^2, x^3, x^4, x^m, expressed in powers of x, and the law of their coefficients.

$\Delta . x^2 = 2\,x + 1$, $\Delta^2 . x^2 = 2$,

$\Delta . x^3 = 3\,x^2 + 3\,x + 1$, $\Delta^2 . x^3 = 6\,x + 6$, $\Delta^3 . x^3 = 6$,

$\Delta . x^4 = 4\,x^3 + 6\,x^2 + 4\,x + 1$, $\Delta^2 . x^4 = 12\,x^2 + 24\,x + 14$,

$\Delta^3 . x^4 = 24\,x + 36$, $\Delta^4 . x^4 = 24$,

$$\Delta^n . x^m = \Delta^n o^m + \frac{m}{1} \Delta^n o^{m-1} . x + \frac{m\,(m-1)}{1 . 2} \Delta^n o^{m-2} . x^2 + \ldots$$

$$\ldots\ldots\ldots + \frac{m\,(m-1) \ldots (n+1)}{1 . 2 \ldots (m-n)} \Delta^n o^n . x^{m-n}.$$

and most extensive views it opens of the nature of analytical operations seem to authorize its universal adoption, not to mention the real inconvenience which more than one author of eminence has been put to for want of some notation founded on principle to express any inverse function without introducing a new characteristic.

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$$\sin 2\,n\,\theta = \frac{n}{1}\,p . \cos \theta - \frac{n\,(n-1)}{1 . 2}\,p^2 . \sin 2\,\theta$$

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$\Delta^3 . x^4 = 24\,x + 36$, $\Delta^4 . x^4 = 24$,

$$\Delta^n . x^m = \Delta^n o^m + \frac{m}{1}\Delta^n o^{m-1} . x + \frac{m\,(m-1)}{1 . 2}\Delta^n o^{m-2} . x^2 + \ldots$$

$$\ldots\ldots\ldots + \frac{m\,(m-1) \ldots . (n+1)}{1 . 2 \ldots . (m-n)}\Delta^n o^n . x^{m-n}.$$

this last equation may be derived from the value of $\Delta^n . x^m$ given in the Appendix, Art. 350, by developing its several terms by the binomial theorem, and collecting the coefficients of similar powers of x. If we then call to mind the definition given in that article of the expression $\Delta^n o^m$ the truth of the above equation will be apparent. But the most regular as well as the easiest mode of obtaining it, is from the general theorem of Art. 361.

(32). To prove the truth of the following theorems

$$(\sin \theta)^n . \cos n\theta =$$

$$\theta^n . \left\{ 1 - \frac{\Delta^n o^{n+2}}{1.2....(n+2)} (2\theta)^2 + \frac{\Delta^n o^{n+4}}{1....(n+4)} (2\theta)^4 - \&c. \right\}$$

$$(\sin \theta)^n . \sin n\theta =$$

$$\theta^n . \left\{ \frac{\Delta^n o^{n+1}}{1....(n+1)} (2\theta) - \frac{\Delta^n o^{n+3}}{1....(n+3)} (2\theta)^3 + \&c. \right\}$$

these may also be deduced from the same general expression for $\Delta^n u_x$ by substituting for u_x, $\sin x\theta$ and $\cos x\theta$, and replacing $\Delta^n u_x$ by the value given in Example (3).

The numbers comprised in the form $\Delta^n o^m$ enter so extensively into the theory of series, and afford such remarkable facilities in the developement of functions that we shall take occasion to annex in this place a short table of their values as far as $\Delta^{10} o^{10}$ and hereafter to present the most remarkable of their properties, sufficient to explain the manner of their employment with reference to this object.

(33). To calculate the actual numerical values of $\Delta^n o^m$ for all values of m and n from 1 up to 10.

	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8	Δ^9	Δ^{10}
0^1	1									
0^2	1	2								
0^3	1	6	6							
0^4	1	14	36	24						
0^5	1	30	150	240	120					
0^6	1	62	540	1560	1800	720				
0^7	1	126	1806	8400	16800	15120	5040			
0^8	1	254	5796	40824	126000	191520	141120	40320		
0^9	1	510	18150	186480	834120	1905120	2328480	1451520	362880	
0^{10}	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800

The inspection of this table affords room for one or two remarks. The value of the function $\Delta^n o^m$ increases with the indices n and m, but is more influenced by the latter than the former. The expressions $\Delta^n o^n$, $\Delta^n o^{n+1}$, $\Delta^n o^{n+2}$, &c. go on perpetually increasing in a very high ratio, and end in surpassing any assignable number. Their rate of increase too surpasses at last that of any assignable geometric progression, as we shall soon see.

(34). To assign the approximate values of $\Delta^n o^n$, $\Delta^n o^{n+1}$, $\Delta^n o^{n+2}$, &c. when n is a very high number.

This may easily be done, if we actually develope $(e^t-1)^n$, or

$$t^n\left\{1+\left(\frac{t^2}{1\,.\,2}+\frac{t^3}{1\,.\,2\,.\,3}+\&c.\right)\right\}^n$$

by the binomial theorem, and compare the coefficients of the powers of t so produced with those of the same powers in the series,

$$t^n\left\{\frac{\Delta^n o^n}{1\ldots\ldots n}+\frac{\Delta^n o^{n+1}}{1\ldots\ldots(n+1)}t+\&c.\right\};\text{ Art. 361. App.}$$

for we thus obtain

$$\Delta^n o^n = 1\,.\,2\ldots\ldots n,$$

$$\Delta^n o^{n+1} = 1\,.\,2\ldots\ldots(n+1)\times\frac{n}{2},$$

$$\Delta^n o^{n+2} = 1\,.\,2\ldots\ldots(n+2)\times\frac{3n^2+n}{24},$$

$$\Delta^n o^{n+3} = 1\,.\,2\ldots\ldots(n+3)\times\frac{n^3+n^2}{48},$$

$$\Delta^n o^{n+4} = 1\,.\,2\ldots\ldots(n+4)\times\frac{15n^4+30n^3+5n^2-2n}{5760},\ \&c.$$

As n increases, these functions therefore increase ultimately at the same rate with those progressions spoken of in the Appendix under the name of hyper-geometrical series (Art. 414.) and when n is very large, we get, by applying the formulæ of Art. 411.

$$\Delta^n 0^n = \sqrt{(2\pi)} \,.\, n^{\frac{1}{2}} \,.\, \left(\frac{n}{e}\right)^n$$

$$\Delta^n 0^{n+1} = \frac{\sqrt{2\pi} \,.\, n^{\frac{3}{2}}}{2} \,.\, \left(\frac{n+1}{e}\right)^{n+1}$$

$$\Delta^n 0^{n+2} = \frac{\sqrt{(2\pi)} \,.\, n^{\frac{5}{2}}}{8} \,.\, \left(\frac{n+2}{e}\right)^{n+2}; \text{ \&c.}$$

or, if we consider that when n is a very high number,

$$(n+1)^n = e \,.\, n^n,\ (n+2)^n = e^2 \,.\, n^n, \text{ \&c.}$$

$$\Delta^n 0^n = \sqrt{(2\pi)} \,.\, n^{\frac{1}{2}} \,.\, \left(\frac{n}{e}\right)^n$$

$$\Delta^n 0^{n+1} = \frac{\sqrt{(2\pi)}}{2} \,.\, n^{\frac{5}{2}} \left(\frac{n}{e}\right)^n$$

$$\Delta^n 0^{n+2} = \frac{\sqrt{(2\pi)}}{8} \,.\, n^{\frac{9}{2}} \,.\, \left(\frac{n}{e}\right)^n$$

$$\Delta^n 0^{n+3} = \frac{\sqrt{(2\pi)}}{48} \,.\, n^{\frac{13}{2}} \,.\, \left(\frac{n}{e}\right)^n; \text{ \&c.}$$

If more exact formulæ be required, the series of powers of $\frac{1}{n}$ must be taken into the account.

(35). To shew that

$$u_{x+n} = u_x + \frac{n}{1} \Delta u_{x-1} + \frac{n(n+1)}{1 \,.\, 2} \Delta^2 u_{x-2} + \frac{n(n+1)(n+2)}{1 \,.\, 2 \,.\, 3} \Delta^3 u_{x-3} + \text{\&c.}$$

The generating function of u_x being $\phi(t)$, that of u_{x+n} is $\frac{1}{t^n}\phi(t)$. Let this be thrown into the form

$$\left\{1 - t\left(\frac{1}{t} - 1\right)\right\}^n . \phi(t)$$

and developed by the binomial theorem. If we then redescend from the generating functions to the coefficients of t^x in their developements, the theorem in question will result.

(36). To prove in general that

$$u_{x+n} = u_x + \frac{n}{1}\Delta u_{x-r} + \frac{n(n+2r-1)}{1.2}\Delta^2 u_{x-2r} +$$
$$+ \frac{n(n+3r-1)(n+3r-2)}{1.2.3}\Delta^3 u_{x-3r} + \&c.$$

The generating function $\frac{1}{t^n}\phi(t)$ of u_{x+n} must be transformed into a series of terms of the form

$$\left\{t^r\left(\frac{1}{t} - 1\right)\right\}^i . \phi(t),$$

and their coefficients shewn to coincide with those of the proposed series. To develope t^{-n} in powers of $t^r\left(\frac{1}{t} - 1\right)$ put the latter function equal to z and we have to develope $\left(\frac{1}{t}\right)^n$ in powers of z, $\frac{1}{t}$ being a function of z given by the equation

$$t^r\left(\frac{1}{t} - 1\right) = z$$

$$\text{or } \frac{1}{t} = 1 + z . \left(\frac{1}{t}\right)^r,$$

Lagrange's theorem demonstrated in Note E enables us to do this. If we put y for $\frac{1}{t}$, we have

$$1 - y + zy^r = 0$$

which gives

$$y^n = \frac{1}{t^n} = 1 + \frac{n}{1}z + \frac{n(n+2r-1)}{1.2}z^2 + \&c.$$

whence the theorem in question results.

SECTION II.

Exercises in the resolution of Functions into Factorials, to prepare them for Integration.

(1). To resolve x^2, x^3, x^4, x^n, into products of the factors x, $x-1$, $x-2$, &c., or as it is sometimes expressed, to reduce them to x and its *preceding values*,

$$x^2 = x + x(x-1)$$

$$x^3 = x + 3x(x-1) + x(x-1)(x-2)$$

$$x^4 = x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3)$$

$$x^n = \frac{\Delta o^n}{1}x + \frac{\Delta^2 o^n}{1.2}x(x-1) + \frac{\Delta^3 o^n}{1.2.3}x(x-1)(x-2) + \&c.$$

The general expression may be deduced from the equation

$$u_{y+x} = u_y + \frac{x}{1}\Delta u_y + \frac{x(x-1)}{1.2}\Delta^2 u_y + \&c.$$

(See Appendix, Art. 345), by making $u_y = y^n$ and then sup-

The generating function of u_x being $\phi(t)$, that of u_{x+n} is $\frac{1}{t^n}\phi(t)$. Let this be thrown into the form

$$\left\{1 - t\left(\frac{1}{t} - 1\right)\right\}^{n} . \phi(t)$$

and developed by the binomial theorem. If we then re-descend from the generating functions to the coefficients of t^x in their developements, the theorem in question will result.

(36). To prove in general that

$$u_{x+n} = u_x + \frac{n}{1}\Delta u_{x-r} + \frac{n(n+2r-1)}{1 \cdot 2}\Delta^2 u_{x-2r} +$$

$$+ \frac{n(n+3r-1)(n+3r-2)}{1 \cdot 2 \cdot 3}\Delta^3 u_{x-3r} + \&c.$$

The generating function $\frac{1}{t^n}\phi(t)$ of u_{x+n} must be transformed into a series of terms of the form

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(See Appendix, Art. 345), by making $u_y = y^n$ and then supposing $y=0$. The values of Δo^n, $\Delta^2 o^n$, &c. when n does not exceed 10 may be taken from the table given in page (9).

(2). To resolve x, x^2, x^3, x^4, x^n into products of the successive factors $x+1$, $x+2$, &c. or to reduce them to *succeeding values* of x.

$$x = -1 + (x+1)$$

$$x^2 = +1 - 3(x+1) + (x+1)(x+2)$$

$$x^3 = -1 + 7(x+1) - 6(x+1)(x+2) + (x+1)(x+2)(x+3)$$

$$x^4 = +1 - 15(x+1) + 25(x+1)(x+2) - 10(x+1)\ldots(x+3) + \\ + (x+1)\ldots\ldots(x+4)$$

$$x^n = (-1)^n \left\{ \frac{\Delta o^{n+1}}{1} - \frac{\Delta^2 o^{n+1}}{1\,.\,2}(x+1) \right. \\ \left. + \frac{\Delta^3 o^{n+1}}{1\,.\,2\,.\,3}(x+1)(x+2) - \&c. \right\}$$

This may be deduced from the same general expression for u_{y+x} by putting $-x$ for x and supposing $u_y = y^{n+1}$ and then proceeding as above.

(3). It has been remarked (App. Art. 370.) that "to keep the numerical coefficients as low as possible in these reductions to the form of factorials is an object of importance," and that "this may be done by a proper disposition of the preceding and succeeding factors." The following examples will shew how this is to be performed.

$$x^3 = (x-1)\,x\,(x+1) + x.$$

$$x^5 = (x-2)(x-1)\,x\,(x+1)(x+2) + 5(x-1)\,x\,(x+1) + x.$$

$$x^7 = (x-3)\ldots(x+3) + 14(x-2)\ldots(x+2) + \\ + 21(x-1)\,x\,(x+1) + x.$$

In these instances it will be observed that no factorials with an even number of factors enter into the expression.

This is a simplification of considerable moment, and that it takes place in general for x^{2n+1} may be demonstated without difficulty as follows.

(4). To resolve x^{2n+1} in the same manner, and to determine the law of the coefficients.

Assume $x^2 = v$. Then $x^{2n+1} = v^n x$, suppose now

$$v^n = A_0 + A_1 (v-1) + A_2 (v-1)(v-4) +$$
$$+ A_3 (v-1)(v-4)(v-9) + \ldots . A_n (v-1)(v-4)..(v-n^2)$$

This assumption is *possible* because the second member is a rational integral function of v of the nth degree, and being reduced to powers of v, and compared with v^n will afford $n+1$ equations of the first degree for the determination of the indeterminate coefficients $A_0, A_1, \ldots . A_n$. The following is however a readier and more elegant method. The above equation being identical in v must hold good whatever numbers are substituted for v, hence if for v we write in succession 1, 4, 9, 25, &c., we get

$$1^{2n} = A_0,$$
$$2^{2n} = A_0 + (2^2 - 1^2) A_1,$$
$$3^{2n} = A_0 + (3^2 - 1^2) A_1 + (3^2 - 1^2)(3^2 - 2^2) A_2$$
$$\&c. = \&c.$$

Whence we derive

$$A_0 = 1^{2n},$$
$$A_1 = \frac{2^{2n}}{2^2 - 1^2} + \frac{1^{2n}}{1^2 - 2^2},$$
$$A_2 = \frac{3^{2n}}{(3^2-1^2)(3^2-2^2)} + \frac{2^{2n}}{(2^2-1^2)(2^2-3^2)} + \frac{1^{2n}}{(1^2-2^2)(1^2-3^2)},$$

and so on, the law being evident and the general value of A_r susceptible of direct expression in functions of x.

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Such then being the values of A_0, A_1, &c., we have

$$x^{2n+1}=xv^n=x\{A_0+A_1(x^2-1^2)+A_2(x^2-1^2)(x^2-2^2)+\&c.\}$$

$$=A_0x+A_1(x-1)x(x+1)+\&c.$$

v being equal to x^2.

The law observed by the coefficients is not a little remarkable. It extends too with a slight modification to cases of much greater generality, and it will hardly be thought irrelevant to the present subject to propose and resolve the following problem.

(5). To develope $F(x)$ in a series of factorial terms of the following form

$$F(x)=A_0+A_1(x-f_1)+A_2(x-f_1)(x-f_2)+\&c.$$

$F(x)$ being any function whatever of x, and f_1, f_2, &c. particular values of any other function f_x corresponding to the values 1, 2, &c. of x.

A process exactly similar to the foregoing, viz., the substitution of f_1, f_2, f_3, &c. in the succession for x and the determination of the coefficients one from another by means of the equations thence arising gives $A_0=F(f_1)$ or, for brevity's sake omitting the parentheses

$$A_0=Ff_1,$$

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$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$A_{n-1}=\frac{Ff_1}{(f_1-f_2)(f_1-f_3)..(f_1-f_n)}+\frac{Ff_2}{(f_2-f_1)(f_2-f_3)...(f_2-f_n)}+\cdots$$

$$\cdots\cdots+\frac{Ff_n}{(f_n-f_1)(f_n-f_2)\cdots(f_n-f_{n-1})}.$$

Several consequences follow from this theorem; 1st, If $F(x)$ be any rational integral function of x of the n^{th} degree, and f_x any function of x whatever, it is easily seen that all the values of the general expression for the coefficients after A_n must vanish of themselves, giving $A_{n+1}=0$, $A_{n+2}=0$, &c. to infinity, which is one of the most general and singular properties of rational integral functions: (2dly), If $F(x)=x^n$ and $f(x)=x^2$, we get the series of coefficients investigated in the last question: (3dly), If $F(x)=$ any rational integral function of x, and $f(x)=x^2$ we obtain a set of coefficients proper for the resolution of such a function as

$$x \,.\, F(x^2)$$

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These instances will suffice to shew that this mode of developing $F(x)$ is not a mere matter of idle speculation. Other and still more extensive applications of it will shortly appear.

(6). To resolve x^2, x^4, x^6, x^{2n}, the even powers of x into factorials where the preceding and succeeding factors occur symmetrically.

This cannot be done, it is evident, by resolving them into factors x, $x \pm 1$, $x \pm 2$, &c. because the degree of any expression such as

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. . . .

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$$A_0 x + A_1 (x-1)\, x\, (x+1) + \text{\&c.}$$

must necessarily be odd. The object may however be accomplished by taking $x \pm \frac{1}{2}$, $x \pm \frac{3}{2}$, &c. for the factors, when we shall find

$$x^2 = \frac{1}{4} \{ 1 + (2x-1)(2x+1) \},$$

$$x^4 = \frac{1}{16} \{ 1 + 10(2x-1)(2x+1)$$
$$+ (2x-3)(2x-1)(2x+1)(2x+3) \}$$

$$x^6 = \frac{1}{64} \{ 1 + 91(2x-1)(2x+1) + 35(2x-3)\ldots(2x+3) +$$
$$+ (2x-5)\ldots\ldots(2x+5) \}.$$

For x^{2n};—put $(2x)^2 = v$, and supposing $F(v) = v^n$, and $f(v) = (2v-1)^2$, we have

$$f_1 = 1^2, f_2 = 3^2, f_3 = 5^2, \text{ \&c.}$$
$$Ff_1 = 1^{2n}, Ff_2 = 3^{2n}, Ff_3 = 5^{2n}, \text{ \&c.},$$

and we shall therefore have by the general theorem above demonstrated (Art. 5. Sect. 2.)

$$v^n = A_0 + A_1(v-1^2) + A_2(v-1^2)(v-3^2) + \text{\&c}; \text{ where}$$
$$A_0 = 1^{2n},$$

$$A_1 = \frac{1^{2n}}{1^2-3^2} + \frac{3^{2n}}{3^2-1^2},$$

$$A_2 = \frac{1^{2n}}{(1^2-3^2)(1^2-5^2)} + \frac{3^{2n}}{(3^2-1^2)(3^2-5^2)} + \frac{5^{2n}}{(5^2-1^2)(5^2-3^2)}.$$

Now since $(2x)^2 = v$, $x^{2n} = \frac{v^n}{2^{2n}}$, and we therefore have since

$$v - 1^2 = \{ \surd(v) - 1 \} \{ \surd(v) + 1 \} = (2x-1)(2x+1),$$

and so on, the following final result, where A_0, &c. have the values above written

$$x^{2n}=\frac{1}{2^{2n}}\{A_0+A_1(2x-1)(2x+1)$$

$$+A_2(2x-3)\ldots(2x+3)+\&c.\}.$$

In the same way may such a function as $F(x^2)$, any *even* function of x be treated.

(7). To complete the factorials

$$x(x+1)(x+3),\text{ and }(2x-1)(2x+1)(2x+5).$$

N. B. By completing them is meant reducing them to the others in which the terms follow the order of an arithmetical progression.

They are respectively equal to

$$x(x+1)+x(x+1)(x+2),\text{ and}$$

$$2(2x-1)(2x+1)+(2x-1)(2x+1)(2x+3).$$

(8). Reduce $(2x+3)^3$ to preceding values and $(x+a)^2$ to succeeding ones of x.

$$(2x-1)(2x-3)(2x-5)+18(2x-1)(2x-3)+$$

$$+76(2x-1)+64=(2x+3)^3$$

$$(x+a)^2=(a-1)^2+(2a-3)(x+1)+(x+1)(x+2).$$

(9). Reduce $x(x^2+2)^2$ to factorials in which the preceding and succeeding factors occur symmetrically. The application of the theorem in (Art. 5. Sect. 2.) gives

$$x(x^2+2)^2=9x+9(x-1)(x)(x+1)$$

$$+(x-2)(x-1)x(x+1)(x+2).$$

(10). Reduce $(x^3+x)(2x^2-6)^3$ in the same manner

$$(x^3+x)(2x^2-6)^3=-128x+56(x-1)x(x+1)$$

$$+ 424 (x-2)(x-1)x(x+1)(x+2)$$

$$+ 176 (x-3)(x-2)(x-1)x(x+1)(x+2)(x+3(+ + (x-4)\ldots\ldots(x+4).$$

(11). Resolve $(x^4+1)^2$ into the smallest possible number of complete factorials.

$$(x^4+1)^2 = \frac{1}{2^8}\{289 + 1140(2x-1)(2x+1)$$

$$+ 998(2x-3)(2x-1)(2x+1)(2x+3)$$

$$+84(2x-5)\ldots(2x+5) + (2x-7)\ldots(2x+7)\};$$

(12). Affect $u_{x+1} \cdot u_{x+3}$ with u_x and reduce it to succeeding values of u_x. *N.B.* By affecting it with u_x is meant introducing u_x as one of the factors of the result. In this and the following examples u_x is understood to have its difference constant, so that

$$u_x = a + hx, \ \Delta u_x = h.$$

Then, $u_{x+1} \cdot u_{x+3} = u_x u_{x+1} + 3 h u_x$.

(13). Affect $u_{x-3} \cdot u_{x+2}$ with u_x

$$u_x u_{x+1} - 2h \cdot u_x + 6h^2 = u_{x-3}\, u_{x+2}.$$

(14). Reduce $(u_{x-1})^2 \cdot (u_{x+1})^2$ to succeeding values of u_x,

$$(u_{x-1})^2 \cdot (u_{x+1})^2 =$$

$$u_{x+1} \ldots u_{x+4} - 10h \cdot u_{x+1} \ldots u_{x+3}$$

$$+ 23h^2 \cdot u_{x+1} u_{x+2} - 9h^3 \cdot u_{x+1}.$$

(15). Affect $u_{x-1}\, u_{x+2}\, u_{x+3}$ with u_x and reduce it to factorials consisting of u_x and its succeeding values.

Ans. $u_x u_{x+1} u_{x+2} + h \cdot u_x u_{x+1} - 2h^2 \cdot u_x - 6h^3$.

(16). Reduce $u_x\, u_{x+1}\, u_{x+2}\, u_{x+4}\, u_{x+5}$, to complete factorials.

Ans. $u_x \ldots u_{x+4} + 2\,h\,u_x \ldots u_{x+3} + 2\,h^2 u_x \ldots u_{x+2}$.

(17). Reduce $u_x \ldots u_{x+n-1} \times u_{x+n+1} \ldots u_{x+m}$ to complete factorials.

$$u_x \ldots u_{x+m-1} + (m-n)\,h\,.\,u_x \ldots u_{x+m-2}$$
$$+ (m-n)\,(m-n-1)\,h^2\,.\,u_x \ldots u_{x+m-3}$$
$$+ \ldots (m-n) \ldots 3\,.\,2\,.\,1\,h^{m-n}\,u_x \ldots u_{x+n-1}.$$

(18). Affect $u_{x-2}\, u_{x-1}\, u_{x+1}$ with u_x.

$$u_x u_{x+1} u_{x+2} - 5\,h\,u_x\,u_{x+1} + 2\,h^2\,u_x + 2\,h^3.$$

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$$- \frac{(n+1)\,n\,(n-1) \times m\,(m+1)\,(m+2)}{1\,.\,2\,.\,3}\,h^3\,.\ \&c. + \&c.$$

The investigation is easy if we employ the principle explained in (Prob. 4. Sect. 2.) by assuming

$$u_x \ldots u_{x+n} = A\, u_{x+m} \ldots u_{x+m+n}$$
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and making u_{x+m+1}, u_{x+m+2}, &c. vanish in succession, which will give so many equations for determining A, B, &c. one from the other. The two last expressions are from Emerson's Increments.

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$$u_{x+\alpha}\, u_{x+\beta} = u_x\, u_{x+1} + (\alpha + \beta - 1)\, h\, u_x + \alpha\beta\,.\,h^2.$$

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where the number of factors is n to a series of complete factorials commencing with u_x and proceeding according to the succeeding values u_{x+1}, u_{x+2}, &c. u_x being as before equal to $a + hx$.

Assume $A_0 + A_1 u_x + A_2\, u_x\, u_{x+1}$ + &c. for the series and make x in succession $-\frac{a}{h}$, $-\left(\frac{a}{h} + 1\right)$, $-\left(\frac{a}{h} + 2\right)$, &c. so as to cause the factors u_x, u_{x+1}, &c. to vanish in succession, and the resulting equation will give the coefficients in succession just as in (Prob. 4. Sect. 2.) Or we may proceed by supposing $v = hx$, $F(v) = u_{x+\alpha} \cdot u_{x+\beta}$, &c.

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Let $S_0 = 1$,

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Then will

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$$+ \frac{h^{n-3}}{1 \cdot 2 \cdot 3} u_x u_{x+1} u_{x+2} \{ \Delta^3 o^3 \cdot S_{n-3} - \&c. \} + \&c.$$

The terms $\Delta o^0 \cdot S_n$ in the coefficient of $\frac{h^{n-1}}{1} u_x$, $\Delta^2 o^0 \cdot S_n - \Delta^2 o^1 \cdot S_{n-1}$ in that of $\frac{h^{n-2}}{1 \cdot 2} u_x u_{x+1}$ and so on being of themselves equal to zero are for brevity omitted in the respective series within the brackets, to which they belong. Nevertheless to preserve the symmetry of the equation, they ought, if not set down, at least to be understood, being in-

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$$+ \frac{h^{n-2}}{1 . 2} u_x u_{x+1} \{ \Delta^2 o^2 . S_{n-2} - \Delta^2 o^3 . S_{n-3} + \&c. \}$$

$$+ \frac{h^{n-3}}{1 . 2 . 3} u_x u_{x+1} u_{x+2} \{ \Delta^3 o^3 . S_{n-3} - \&c. \} + \&c.$$

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Some particular cases of the general theorem deserve to be stated separately, as they afford transformations which which we may have occasion to use.

(25). To resolve u^n_{x+1} into succeeding factors affected with u_x. Here $\alpha = \beta = \&c. = 1$: and

$$u^n_{x+1} = h^n + \frac{h^{n-1}}{1} u_x \times$$

$$\left\{ \frac{n}{1} \Delta o - \frac{n(n-1)}{1.2} \Delta o^2 + \frac{n(n-1)(n-2)}{1.2.3} \Delta o^3 - \&c. \right\}$$

$$+ \frac{h^{n-2}}{1.3} u_x u_{x+1} \times$$

$$\left\{ \frac{n(n-1)}{1.2} \Delta^2 o^2 - \frac{n(n-1)(n-2)}{1.2.3} \Delta^2 o^3 - \&c. \right\}$$

$$+ \frac{h^{n-3}}{1.2.3} u_x u_{x+1} u_{x+2} \times$$

$$\left\{ \frac{n(n-1)(n-2)}{1.2.3} \Delta^3 o^3 - \&c. \right\} + \&c.$$

But this is not the simplest or most elegant form in which this equation can be expressed. If we complete the series within the brackets by inserting the deficient terms at their commencement, and then separate the symbols of operation from those of quantity they will become

$$- \Delta \left\{ 1 - \frac{n}{1} o + \frac{n(n-1)}{1.2} o^2 - \&c. \right\} = - \Delta (1-0)^n$$

$$+ \Delta^2 \left\{ 1 - \frac{n}{1} o + \frac{n(n-1)}{1.2} o^2 - \&c. \right\} = + \Delta^2 (1-0)^n, \&c.$$

Now the reader will find it demonstrated in Art. 17. Sect. 7. that the expressions $\Delta(1-0)^n$, $\Delta^2(1-0)^n$, &c. are respectively equivalent to

$$(-1)^n . \{ \Delta o^n - \Delta^2 o^n + \Delta^3 o^n \ldots . \pm \Delta^n o^n \},$$
$$(-1)^n . \{ \Delta^2 o^n - \Delta^3 o^n + \ldots . \mp \Delta^n o^n \},$$
$$(-1)^n . \{ \Delta^3 o^n \ldots . \pm \Delta^n o^n \}, \text{ \&c.}$$

or, inverting the order of writing their terms, to

$$-\Delta^n o^n + \Delta^{n-1} o^n \ldots . \pm \Delta o^n$$
$$+\Delta^n o^n - \Delta^{n-1} o^n \ldots . \pm \Delta^2 o^n, \text{ \&c.}$$

so that by substitution we shall have

$$u^n_{x+1} = h^n + \frac{\Delta^n o^n - \Delta^{n-1} o^n + \ldots . \pm \Delta o^n}{1} h^{n-1} u_x$$
$$+ \frac{\Delta^n o^n - \Delta^{n-1} o^n + \ldots . \mp \Delta^2 o^n}{1.2} h^{n-2} u_x u_{x+1}$$
$$+ \frac{\Delta^n o^n - \Delta^{n-1} o^n + \ldots . \pm \Delta^3 o^n}{1.2.3} h^{n-3} u_x u_{x+1} u_{x+2}$$
$$+ \text{ \&c.}$$

The reader must not be startled by the employment of o as an algebraic symbol in such expressions as $\Delta(1-0)^n$. He will call to mind that this and similar expressions are mere abbreviations and have no meaning beyond what is expressed by their developement. The transformation in the latter part of this problem cannot, however, be comprehended without a previous knowledge of those more general properties of the functions $\Delta^m o^n$ which will be hereafter demonstrated, and is only inserted in this place that things relating to the same subject may be kept together.

(26). To resolve u^n_x into u_x and succeeding factors

$$u_x^n = (-1)^{n+1} \left\{ \frac{\Delta o^n}{1} h^{n-1} u_x - \frac{\Delta^2 o^n}{1.2} h^{n-2} u_x u_{x+1} + \text{\&c.} \right\}; \ldots (a.)$$

must necessarily be odd. The object may however be accomplished by taking $x \pm \frac{1}{2}$, $x \pm \frac{3}{2}$, &c. for the factors, when we shall find

$$x^2 = \frac{1}{4}\{1+(2x-1)(2x+1)\},$$

$$x^4 = \frac{1}{16}\{1+10(2x-1)(2x+1)$$

$$+(2x-3)(2x-1)(2x+1)(2x+3)\}$$

$$x^6 = \frac{1}{64}\{1+91(2x-1)(2x+1)+35(2x-3)\ldots(2x+3)+$$

$$+(2x-5)\ldots\ldots(2x+5)\}.$$

For x^{2n};—put $(2x)^2 = v$, and supposing $F(v) = v^n$, and $f(v)=(2v-1)^2$, we have

$$f_1 = 1^2,\ f_2 = 3^2,\ f_3 = 5^2,\ \&c.$$

$$Ff_1 = 1^{2n},\ Ff_2 = 3^{2n},\ Ff_3 = 5^{2n},\ \&c.,$$

and we shall therefore have by the general theorem above demonstrated (Art. 5. Sect. 2.)

$$v^n = A_0 + A_1(v-1^2) + A_2(v-1^2)(v-3^2) + \&c;\ \text{where}$$

$$A_0 = 1^{2n},$$

$$A_1 = \frac{1^{2n}}{1^2-3^2} + \frac{3^{2n}}{3^2-1^2},$$

$$A_2 = \frac{1^{2n}}{(1^2-3^2)(1^2-5^2)} + \frac{3^{2n}}{(3^2-1^2)(3^2-5^2)} + \frac{5^{2n}}{(5^2-1^2)(5^2-3^2)}.$$

Now since $(2x)^2 = v$, $x^{2n} = \frac{v^n}{2^{2n}}$, and we therefore have since

$$v-1^2 = \{\sqrt{(v)}-1\}\{\sqrt{(v)}+1\} = (2x-1)(2x+1),$$

and so on, the following final result, where A_0, &c. have the values above written

$$x^{2n}=\frac{1}{2^{2n}}\{A_0+A_1(2x-1)(2x+1)$$

$$+A_2(2x-3)\ldots(2x+3)+\&c.\}.$$

In the same way may such a function as $F(x^2)$, any *even* function of x be treated.

(7). To complete the factorials

$$x(x+1)(x+3), \text{ and } (2x-1)(2x+1)(2x+5).$$

N. B. By completing them is meant reducing them to the others in which the terms follow the order of an arithmetical progression.

They are respectively equal to

$$x(x+1)+x(x+1)(x+2), \text{ and}$$

$$2(2x-1)(2x+1)+(2x-1)(2x+1)(2x+3).$$

(8). Reduce $(2x+3)^3$ to preceding values and $(x+a)^2$ to succeeding ones of x.

$$(2x-1)(2x-3)(2x-5)+18(2x-1)(2x-3)+$$

$$+76(2x-1)+64=(2x+3)^3$$

$$(x+a)^2=(a-1)^2+(2a-3)(x+1)+(x+1)(x+2).$$

(9). Reduce $x(x^2+2)^2$ to factorials in which the preceding and succeeding factors occur symmetrically. The application of the theorem in (Art. 5. Sect. 2.) gives

$$x(x^2+2)^2=9x+9(x-1)(x)(x+1)$$

$$+(x-2)(x-1)x(x+1)(x+2).$$

(10). Reduce $(x^3+x)(2x^2-6)^3$ in the same manner

$$(x^3+x)(2x^2-6)^3=-128x+56(x-1)x(x+1)$$

$$+ 424 (x-2)(x-1)x(x+1)(x+2)$$

$$+ 176(x-3)(x-2)(x-1)x(x+1)(x+2)(x+3(+ + (x-4)\ldots$$

$$\ldots(x+4).$$

(11). Resolve $(x^4+1)^2$ into the smallest possible number of complete factorials.

$$(x^4+1)^2 = \frac{1}{2^8}\{289 + 1140(2x-1)(2x+1)$$

$$+ 998(2x-3)(2x-1)(2x+1)(2x+3)$$

$$+ 84(2x-5)\ldots(2x+5) + (2x-7)\ldots(2x+7)\}.$$

(12). Affect $u_{x+1} \cdot u_{x+2}$ with u_x and reduce it to succeeding values of u_x. *N.B.* By affecting it with u_x is meant introducing u_x as one of the factors of the result. In this and the following examples u_x is understood to have its difference constant, so that

$$u_x = a + hx, \ \Delta u_x = h.$$

Then, $u_{x+1} \cdot u_{x+2} = u_x u_{x+1} + 3h u_x$.

(13). Affect $u_{x-3} \cdot u_{x+2}$ with u_x

$$u_x u_{x+1} - 2h \cdot u_x + 6h^2 = u_{x-3} u_{x+2}.$$

(14). Reduce $(u_{x-1})^2 \cdot (u_{x+1})^2$ to succeeding values of u_x,

$$(u_{x-1})^2 \cdot (u_{x+1})^2 =$$

$$u_{x+1} \ldots u_{x+4} - 10h \cdot u_{x+1} \ldots u_{x+3}$$

$$+ 23h^2 \cdot u_{x+1} u_{x+2} - 9h^3 \cdot u_{x+1}.$$

(15). Affect $u_{x-1}\, u_{x+2}\, u_{x+3}$ with u_x and reduce it to factorials consisting of u_x and its succeeding values.

Ans. $u_x u_{x+1} u_{x+2} + h \cdot u_x u_{x+1} - 2h^2 \cdot u_x - 6h^3$.

(16). Reduce $u_x\, u_{x+1}\, u_{x+2}\, u_{x+4}\, u_{x+5}$, to complete factorials.

Ans. $u_x \ldots u_{x+4} + 2\,h\,u_x \ldots u_{x+3} + 2\,h^2 u_x \ldots u_{x+2}$.

(17). Reduce $u_x \ldots u_{x+n-1} \times u_{x+n+1} \ldots u_{x+m}$ to complete factorials.

$$u_x \ldots u_{x+m-1} + (m-n)\,h\,.\,u_x \ldots u_{x+m-2}$$
$$+ (m-n)\,(m-n-1)\,h^2\,.\,u_x \ldots u_{x+m-3}$$
$$+ \ldots (m-n) \ldots 3\,.\,2\,.\,1\,h^{m-n}\,u_x \ldots u_{x+n-1}.$$

(18). Affect $u_{x-2}\,u_{x-1}\,u_{x+1}$ with u_x.

$$u_x u_{x+1} u_{x+2} - 5\,h\,u_x\,u_{x+1} + 2\,h^2\,u_x + 2\,h^3.$$

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$$u_{x+1} \ldots u_{x+n} = u_x \ldots u_{x+n-1} + n\,h\,u_x \ldots u_{x+n-2} +$$
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$$+ \frac{h^{n-1}}{1} u_x \{ \Delta o \,.\, S_{n-1} - \Delta o^2 \,.\, S_{n-2} + \Delta o^3 \,.\, S_{n-3} - \&c. \}$$

$$+ \frac{h^{n-2}}{1\,.\,2} u_x u_{x+1} \{ \Delta^2 o^2 \,.\, S_{n-2} - \Delta^2 o^3 \,.\, S_{n-3} + \&c. \}$$

$$+ \frac{h^{n-3}}{1\,.\,2\,.\,3} u_x u_{x+1} u_{x+2} \{ \Delta^3 o^3 \,.\, S_{n-3} - \&c. \} + \&c.$$

The terms $\Delta o^0 \,.\, S_n$ in the coefficient of $\frac{h^{n-1}}{1} u_x$, $\Delta^2 o^0 \,.\, S_n - \Delta^2 o^1 \,.\, S_{n-1}$ in that of $\frac{h^{n-2}}{1\,.\,2} u_x u_{x+1}$ and so on being of themselves equal to zero are for brevity omitted in the respective series within the brackets, to which they belong. Nevertheless to preserve the symmetry of the equation, they ought, if not set down, at least to be understood, being included in the law of the other terms, and *this remark is to be considered as applying to all similar cases,* $\Delta^0 o^0$ *being unity and* Δo^0, $\Delta^2 o^0$, $\Delta^2 o^1$, $\Delta^3 o^0$, $\Delta^3 o^1$, $\Delta^3 o^2$, &c. *respectively zero.*

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$$\left\{ \frac{n}{1} \Delta o - \frac{n(n-1)}{1 \, . \, 2} \Delta o^2 + \frac{n(n-1)(n-2)}{1 \, . \, 2 \, . \, 3} \Delta o^3 - \&c. \right\}$$

$$+ \frac{h^{n-2}}{1 \, . \, 3} u_x u_{x+1} \times$$

$$\left\{ \frac{n(n-1)}{1 \, . \, 2} \Delta^2 o^2 - \frac{n(n-1)(n-2)}{1 \, . \, 2 \, . \, 3} \Delta^2 o^3 - \&c. \right\}$$

$$+ \frac{h^{n-3}}{1 \, . \, 2 \, . \, 3} u_x u_{x+1} u_{x+2} \times$$

$$\left\{ \frac{n(n-1)(n-2)}{1 \, . \, 2 \, . \, 3} \Delta^3 o^3 - \&c. \right\} + \&c.$$

But this is not the simplest or most elegant form in which this equation can be expressed. If we complete the series within the brackets by inserting the deficient terms at their commencement, and then separate the symbols of operation from those of quantity they will become

$$- \Delta \left\{ 1 - \frac{n}{1} o + \frac{n(n-1)}{1 \, . \, 2} o^2 - \&c. \right\} = - \Delta (1 - 0)^n$$

$$+ \Delta^2 \left\{ 1 - \frac{n}{1} o + \frac{n(n-1)}{1 \, . \, 2} o^2 - \&c. \right\} = + \Delta^2 (1 - 0)^n, \&c.$$

Now the reader will find it demonstrated in Art. 17. Sect. 7. that the expressions $\Delta(1-0)^n$, $\Delta^2(1-0)^n$, &c. are respectively equivalent to

$$(-1)^n . \{ \Delta o^n - \Delta^2 o^n + \Delta^3 o^n \ldots . \pm \Delta^n o^n \},$$
$$(-1)^n . \{ \Delta^2 o^n - \Delta^3 o^n + \ldots . \mp \Delta^n o^n \},$$
$$(-1)^n . \{ \Delta^3 o^n \ldots . \pm \Delta^n o^n \}, \text{ \&c.}$$

or, inverting the order of writing their terms, to

$$-\Delta^n o^n + \Delta^{n-1} o^n \ldots . \pm \Delta o^n$$
$$+\Delta^n o^n - \Delta^{n-1} o^n \ldots . \pm \Delta^2 o^n, \text{ \&c.}$$

so that by substitution we shall have

$$u^n_{x+1} = h^n + \frac{\Delta^n o^n - \Delta^{n-1} o^n + \ldots . \pm \Delta o^n}{1} h^{n-1} u_x$$
$$+ \frac{\Delta^n o^n - \Delta^{n-1} o^n + \ldots . \mp \Delta^2 o^n}{1 . 2} h^{n-2} u_x u_{x+1}$$
$$+ \frac{\Delta^n o^n - \Delta^{n-1} o^n + \ldots . \pm \Delta^3 o^n}{1 . 2 . 3} h^{n-3} u_x u_{x+1} u_{x+2}$$
$$+ \text{ \&c.}$$

The reader must not be startled by the employment of o as an algebraic symbol in such expressions as $\Delta(1-0)^n$. He will call to mind that this and similar expressions are mere abbreviations and have no meaning beyond what is expressed by their developement. The transformation in the latter part of this problem cannot, however, be comprehended without a previous knowledge of those more general properties of the functions $\Delta^m o^n$ which will be hereafter demonstrated, and is only inserted in this place that things relating to the same subject may be kept together.

(26). To resolve u^n_x into u_x and succeeding factors

$$u_x^n = (-1)^{n+1} \left\{ \frac{\Delta o^n}{1} h^{n-1} u_x - \frac{\Delta^2 o^n}{1 . 2} h^{n-2} u_x u_{x+1} + \text{\&c.} \right\}; \ldots (a.)$$

If in this we put $n+1$ for n and divide by u_x we get as follows:

$$u^n_x = (-1)^n \left\{ \frac{\Delta o^{n+1}}{1} h^n - \frac{\Delta^2 o^{n+1}}{1 \cdot 2} h^{n-1} u_{x+1} \right.$$

$$\left. + \frac{\Delta^3 o^{n+1}}{1 \cdot 2 \cdot 3} h^{n-2} u_{x+1} u_{x+2} - \&c. \right\} ; \ldots . (b).$$

(27). Let $u_x = a + hx$, $u'_x = a' + h'x$. To resolve $(u_x)^n$ into u'_x and succeeding factors, u'_{x+1}, &c.

$$(a+hx)^n = \left(\frac{h}{h'}\right)^n \cdot \left(\frac{ah'}{h} + h'x\right)^n$$

$$= \left(\frac{h}{h'}\right)^n \cdot \left\{(a' + h'x) + h' \cdot \frac{ah' - a'h}{hh'}\right\}^n$$

or, $$(u_x)^n = \left(\frac{h}{h'}\right)^n \cdot (u'_x + \alpha)^n$$

where $$\alpha = \frac{ah' - ha'}{hh'} = \frac{a}{h} - \frac{a'}{h'}.$$

Therefore in the general expression (24) writing u'_x for u_x and making β, γ, &c. equal to α, we get as follows:

$$u_x^n = \left(\frac{h}{h'}\right)^n \left\{ h'^n \cdot \alpha^n \right.$$

$$+ \frac{h'^{n-1}}{1} u'_x \left(\frac{n}{1} \alpha^{n-1} \Delta o - \frac{n(n-1)}{1 \cdot 2} \alpha^{n-2} \Delta o^2 + \&c.\right)$$

$$+ \frac{h'^{n-2}}{1 \cdot 2} u'_x u'_{x+1} \left(\frac{n(n-1)}{1 \cdot 2} \alpha^{n-2} \Delta^2 o^2 - \&c.\right)$$

$$\left. + \&c. \right\}$$

Now, it will be proved in Art. 17. Sect. 7. that the series

$$-\Delta\left\{a^n-\frac{n}{1}a^{n-1}.o+\frac{n(n-1)}{1.2}a^{n-2}.o^2-\&c.\right\},$$

$$+\Delta^2\left\{a^n-\frac{n}{1}a^{n-1}.o+\frac{n(n-1)}{1.2}a^{n-2}.o^2-\&c.\right\}, \&c.$$

which the separation of symbols of operation from those of quantity (as in the last example but one) produces from the coefficients of the several terms $\frac{h'^{n-1}}{1}u_x$, &c. and which have for their abbreviations respectively

$$-\Delta(a-o)^n,\ +\Delta^2(a-o)^n, \&c.$$

are equivalent to other series, the respective abbreviations of which are

$$(-1)^{n+1}\frac{\Delta}{(1+\Delta)^a}o^n,\quad (-1)^{n+2}.\frac{\Delta^2}{(1+\Delta)^a}o^n,$$

$$(-1)^{n+3}.\frac{\Delta^3}{(1+\Delta)^a}o^n, \&c.$$

the series themselves being

$$(-1)^{n+1}\left\{\Delta o^n-\frac{a}{1}\Delta^2 o^n+\frac{a(a+1)}{1.2}\Delta^3 o^n\ldots\ldots\right.$$

$$\left.\pm\frac{a(a+1)\ldots.(a+n-2)}{1.2\ldots.(n-1)}\Delta^n o^n\right\},$$

$$(-1)^{n+2}\left\{\Delta^2 o^n-\frac{a}{1}\Delta^3 o^n\ldots\ldots\right.$$

$$\left.\pm\frac{a(a+1)\ldots.(a+n-3)}{1.2\ldots.(n-2)}\Delta^n o^n\right\}, \&c.$$

and the term a^n itself is shewn by (Art. 17. Sect. 7.) to be equivalent to $(-1)^n.\frac{1}{(1+\Delta)^a}o^n$, or to the series

$$(-1)^n\left\{o^n-\frac{a}{1}\Delta o^n+\ldots.\pm\frac{a(a+1)\ldots.(a+n-1)}{1.2\ldots.n}\Delta^n o^n\right\}.$$

Thus we arrive at the following equation

$$u_x^n = \left(-\frac{h'}{h}\right)^n \cdot \left\{ \frac{1}{(1+\Delta)^a} o^n \cdot h'^n - \frac{\Delta}{(1+\Delta)^a} o^n \cdot \frac{h'^{n-1}}{1} u'_x + \frac{\Delta^2}{(1+\Delta)^a} o^n \cdot \frac{h'^{n-2}}{1 \cdot 2} u'_x u'_{x+1} - \&c. \right\}$$

which affords an indefinite number of different ways of transforming any given expression of this kind. For example:

$$(28).\ (x+a)^n = (-1)^n \cdot \left\{ \frac{1}{(1+\Delta)^a} o^n - \frac{\Delta}{(1+\Delta)^a} o^n \cdot \frac{x}{1} + \frac{\Delta^2}{(1+\Delta)^a} o^n \cdot \frac{x(x+1)}{1 \cdot 2} - \&c. \right\}.$$

SECTION III.

Exercises in the Reduction of Fractional Expressions to Integrable Forms.

(1). Reduce $\frac{x}{(x+1)(x+2)(x+3)}$ to an integrable form.

It becomes

$$\frac{1}{(x+1)(x+2)} - \frac{3}{(x+1)(x+2)(x+3)}.$$

$$(2).\ \frac{1}{(x+1)(x+3)} =$$

$$= \frac{1}{(x+1)(x+2)} - \frac{1}{(x+1)(x+2)(x+3)}.$$

(3). $$\frac{u_{x+1}}{u_x u_{x+2} u_{x+3}} = \frac{1}{u_x u_{x+1}} - \frac{3h}{u_x u_{x+1} u_{x+2}}$$

$$+ \frac{4h^2}{u_x \ldots u_{x+3}} \text{ where } u_x = a + hx.$$

(4). $$\frac{(x+1)^2}{x(x+2)(x+3)(x+4)} = \frac{1}{x(x+1)} - \frac{6}{x(x+1)(x+2)}$$

$$+ \frac{19}{x \ldots\ldots (x+3)} - \frac{27}{x \ldots\ldots (x+4)}.$$

(5). $$\frac{3x-4}{(4x^2-1)(2x-7)} = \frac{3}{2} \cdot \frac{1}{(2x-1)(2x+1)}$$

$$+ \frac{13}{2} \cdot \frac{1}{(2x-3)(2x-1)(2x+1)} + \frac{26}{(2x-5)\ldots\ldots(2x+1)}$$

$$+ \frac{52}{(2x-7)\ldots\ldots(2x+1)}.$$

(6). Theorem (from Emerson's increments,)

$$\frac{1}{u_{x+n}} = \frac{1}{u_x} - \frac{nh}{u_x u_{x+1}} + \frac{n(n-1)h^2}{u_x u_{+1} u_{x+2}} - \&c.$$

This is immediately deducible from (Prob. 21. Sect. 2.) by making $m = 1$, writing $n - 1$ for n, and dividing the whole equation so prepared by $u_x \ldots u_{x+n}$.

(7). A rather more general theorem (given by the same author) is the following:

$$\frac{1}{u_{x+n}} = \frac{1}{u_{x+m}} - \frac{(n-m)h}{u_{x+m} u_{x+m+1}}$$

$$+ \frac{(n-m)(n-m-1)h^2}{u_{x+m} u_{x+m+1} u_{x+m+2}} - \&c.$$

which may be proved by a process almost precisely the same.

These two transformations are not without their use in facilitating reductions of a certain class. For instance,

(8). To reduce $\frac{1}{u_{x-1}\,u_{x+2}}$ to an integrable form,

$$\frac{1}{u_{x+2}} = \frac{1}{u_x} - \frac{2h}{u_x u_{x+1}} + \frac{2h^2}{u_x\, u_{x+1}\, u_{x+2}}$$

therefore

$$\frac{1}{u_{x-1}\,u_{x+2}} = \frac{1}{u_{x-1}\,u_x} - \frac{2h}{u_{x-1}\ldots\ldots u_{x+1}} + \frac{2h^2}{u_{x-1}\ldots\ldots u_{x+2}}.$$

(9). To reduce $\frac{1}{u_x\,.\,u_{x+1}\,.\,u_{x+5}}$ in like manner

$$\frac{1}{u_{x+5}} = \frac{1}{u_{x+2}} - \frac{3h}{u_{x+2}\,.\,u_{x+3}} + \frac{6h^2}{u_{x+2}\,u_{x+3}\,u_{x+4}} - \frac{6h^3}{u_{x+2}\,u_{x+3}\,u_{x+4}\,u_{x+5}},$$

and

$$\frac{1}{u_x\,u_{x+1}\,u_{x+5}} = \frac{1}{u_x\ldots\ldots u_{x+2}} - \frac{3h}{u_x\ldots\ldots u_{x+3}} + \frac{6h^2}{u_x\ldots\ldots u_{x+4}} - \frac{6h^3}{u_x\ldots\ldots u_{x+5}}.$$

SECTION IV.

Exercises in the Integration of Equations of Differences.

(1). To integrate the equation

$$u_{x+1} - (x+1)u_x = 1 . 2 . . . . (x+1).$$

The complete integral is

$$u_x = (C + x) \times 1 . 2 . . . . . . x$$

C being an arbitary constant quantity.

(2). $$u_{x+1} - p\, a^{2x} u_x = q\, a^{x^2}$$

$$u_x = C . p^{x-1} . a^{x^2 - x} + \frac{q\, a^{(x-1)^2}}{1 - ap}.$$

(3). $$(x+1)^2 \{ u_{x+1} - a u_x \} = a^x$$

$$u_x = C . a^x + a^{x-1} . \Sigma \frac{1}{(x+1)^2}$$

$$= C . a^x + a^{x-1} \left\{ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... \frac{1}{x^2} \right\}.$$

In this Example the integration of $\frac{1}{(1+x)^2}$ cannot be performed in finite terms, unless we express it in a series the number of whose terms is variable. This we have done, and in many cases (as we shall see) such a result is useful and satisfactory.

(4). $$u_{x+2} - p\, u_{x+1} + q\, u_x = 0$$

$$u_x = {}^1C \cdot \left\{\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}\right\}^x$$

$$+ {}^2C \cdot \left\{\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}\right\}^x$$

${}^1C$ and ${}^2C$ being two arbitrary constants.

(5). $u_{x+2} - a(x+2)u_{x+1} + b(x+1)(x+2)u_x = 0$

$$u_x = 1 \,.\, 2 \ldots . x \,\{ {}^1C \,.\, a^x + {}^2C \,.\, \beta^x \}$$

a and β being the two roots of $u^2 + a\,u + b = 0$. The integration is performed by assuming

$$u_x = 1 \,.\, 2 \ldots\ldots x \times v_x.$$

(6). $u_{x+2} + a\,p^{2x}\,u_{x+1} + b\,p^{4x}\,u_x = c\,q^x \,.\, p^{x^2}$

$$u_x = p^{x^2+x}\,\{ {}^1C \,.\, a^x + {}^2C \,.\, \beta^x \} + \frac{c\,q^x\,p^{(x-2)^2}}{q^2 + a\,p\,q + b\,p^4}$$

a and β being the roots of

$$u^2 + \frac{a}{p^3}\,u + \frac{b}{p^6} = 0.$$

(7). $u_{x+2} + a \,.\, \phi(x+1)u_{x+1} + b \,.\, \phi(x) \,.\, \phi(x+1)\,u_x = c$, $\phi(x)$ being any function of x whatever;

$$u_x = P\,\phi(x-1) \times$$

$$\left\{ {}^1C\,a^x + {}^2C\,\beta^x + a^{x-1}\,\Sigma \,.\, \frac{\beta^{x-1}}{a^x}\,\Sigma\,\frac{c}{\beta^x \,.\, P\,\phi(x+1)} \right\},$$

$P\,\phi(x)$ denoting $\phi(1) \,.\, \phi(2) \ldots . \phi(x)$, and a and β being roots of $u^2 + a\,u + b = 0$.

(8). $u_{x+3} + a\,p^x\,u_{x+2} + b\,p^{2x}\,u_{x+1} + c\,p^{3x}\,u_x = 0$

$$u_x = p^{\frac{x(x+1)}{2}}\,\{ {}^1C \,.\, a^x + {}^2C \,.\, \beta^x + {}^3C \,.\, \gamma^x \}$$

a, β, γ, being the three roots of

$$u^3 + \frac{a}{p^3}\,u^2 + \frac{b}{p^5}\,u + \frac{c}{p^6} = 0.$$

(9). $u_{x+n}+ap^x u_{x+n-1}+bp^{2x}u_{x+n-2}+\ldots kp^{nx}u_x=0,$

$$u_x=p^{\frac{x(x+1)}{2}}\{{}^1C.\alpha^x+{}^2C.\beta^x+\ldots{}^nC.\nu^x\}$$

$\alpha, \beta, \ldots \nu$ being the n roots of

$$u^n+\frac{a}{p^n}u^{n-1}+\frac{b}{p^{2n-1}}u^{n-2}+\frac{c}{p^{3n-3}}u^{n-3}+\ldots\frac{k}{p^{\frac{n(n+1)}{2}}}=0.$$

(10). $u_{x+2}+au_{x+1}+bu_x=X_x,$

$$u_x={}^1C.\alpha^x+{}^2C.\beta^x+\frac{1}{b}\alpha^x.\Sigma.\left(\frac{\beta}{\alpha}\right)^x\Sigma\frac{X_x}{\beta^x};$$

α and β being the roots of $u^2+au+b=0$.

(11). $\Delta^n u_x=X_x,$

$$u_x={}^1C+{}^2C.x+{}^3C.x^2+\ldots{}^nC.x^{n-1}+\Sigma^n X_x.$$

(12). $\Delta u_x+\Delta^2 u_x=42\,u_x$

$$u_x={}^1C.7^x+{}^2C.(-6)^x.$$

(13). $(x+3)^2.u_{x+2}-2\frac{(x+2)^2}{x+1}u_{x+1}$

$$+\frac{(x+1)^2(x+2)}{x}u_x=0.$$

is to be integrated by the help of a given particular integral $\left(\frac{x}{x+1}=u_x\right)$.

$$u_x={}^1C.\frac{x}{x+1}+{}^2C.\left(\frac{x}{x+1}\right)^2.$$

(14.) $u_{x+1}-2u_x^2+1=0;$

$$u_x=\tfrac{1}{2}\{C^{2^x}+C^{-2^x}\};$$

The substitution to be used is $u_x=\cos v_x$.

(15). $u^2_{x+1} - 4u^2_x(u^2_x + 1) = 0,$

$$u_x = \tfrac{1}{2}\left\{ C^{2^x} - C^{-2^x} \right\}.$$

The substitution is $u_x = \sqrt{(-1)} \cdot \sin v_x$.

(16). $u^2_x - 4u_x u_{x+1} + a(2u_x + u_{x+1}) = 0,$

Substitute $\frac{a}{2} \cdot \frac{2v_x - 1}{v_x + 1}$ for u_x and again, in the resulting equation, $-\cos w_x$ for v_x, when it will assume an integrable form.

(17). $u^2_x + 2u_x u_{x+1} - a(4u_x - u_{x+1}) = 0,$

Substitute $2a \cdot \frac{v_x + 1}{2v_x - 1}$ for u_x

(18). $u_{x+2} u_{x+1} u_x = a(u_{x+2} + u_{x+1} + u_x),$

Assume $u_x = \sqrt{(a)} \cdot \tan v_x$ then will the equation for determining v_x be

$$v_{x+2} + v_{x+1} + v_x = 0,$$

whence finally

$$u_x = \sqrt{(a)} \cdot \tan\left\{ {}^1C \cdot \cos\frac{2\pi x}{3} + {}^2C \cdot \sin\frac{2\pi x}{3} \right\}.$$

(19). The same substitution will be found to succeed in the equations

$$u_{x+2} u_{x+1} u_x + a(u_{x+2} - u_{x+1} + u_x) = 0,$$

$$u_{x+2} u_{x+1} u_x - a(u_{x+2} - u_{x+1} - u_x) = 0,$$

$$u_{x+2} u_{x+1} u_x + a(u_{x+2} + u_{x+1} - u_x) = 0.$$

(20). To integrate the equation

$$u_{x+1} u_x - a(u_{x+1} - u_x) + 1 = 0.$$

Laplace, *Journ de l'Ecole Polytechnique*, Cah. 15.

Differentiate it relative to x, and it becomes

$$(a + u_{x+1})\frac{d\,u_x}{d\,x} - (a - u_x)\frac{d\,u_{x+1}}{d\,x} = 0.$$

But the proposed equation gives

$$a = \frac{u_{x+1}\,u_x + 1}{u_{x+1} - u_x};$$

so that by eliminating a, we find

$$\frac{d\,u_{x+1}}{1 + u^2_{x+1}} - \frac{d\,u_x}{1 + u^2_x} = 0,$$

and integrating

$$\int \frac{d\,u_{x+1}}{1 + u^2_{x+1}} - \int \frac{d\,u_x}{1 + u^2_x} = A$$

A being a certain function of a to be hereafter determined. In fact, since both this and the proposed, are each of them complete integrals of one and the same differential equation, the one can be nothing more than a transformation of the other. Now this latter is equivalent to

$$\Delta \int \frac{d\,u_x}{1 + u^2_x} = A$$

because $\int \frac{d\,u_{x+1}}{1 + u^2_{x+1}}$ is the same function of u_{x+1} that $\int \frac{d\,u_x}{1 + u^2_x}$ is of x. This equation is immediately integrable (relative to the characteristic Δ) and gives

$$\int \frac{d\,u_x}{1 + u^2_x} = A\,x + C$$

C being an arbitrary constant. To determine the function A of a we first have to assign $\int \frac{d\,u_x}{1 + u^2_x}$. Now this is arc $(\tan = u_x)$ or, $\tan^{-1}(u_x)$, hence

$$\tan^{-1} u_x = A x + C$$

$$u_x = \tan (A x + C)$$

therefore,
$$u_{x+1} = \tan (A x + C + A)$$

$$= \frac{u_x + \tan A}{1 - u_x . \tan A}.$$

But the proposed equation gives

$$u_{x+1} = \frac{u_x + \frac{1}{a}}{1 - u_x . \frac{1}{a}},$$

which compared with the foregoing gives $\tan A = \frac{1}{a}$, or $A = \tan^{-1} \frac{1}{a}$. The complete integral then of the proposed equation is

$$u_x = \tan \left\{ x . \tan^{-1} \left(\frac{1}{a}\right) + C \right\}$$

which, although in appearance transcendental, is easily freed from that form, and reduced to a particular case of the integral found in the Appendix. Art. 386. of the more general equation there discussed. The same method applies to certain other equations, for which see the author cited.

(21). To integrate the more general equation

$$u_{x+1} u_x - a_x (u_{x+1} - u_x) + 1 = 0,$$

The proposed equation gives

$$\frac{1}{a_x} = \frac{u_{x+1} - u_x}{1 + u_x u_{x+1}} = \frac{\Delta u_x}{1 + u_x u_{x+1}}$$

but by (Art. 28. Sect. 1.) we have

$$\Delta \tan^{-1} u_x = \tan^{-1} \left(\frac{\Delta u_x}{1 + u_x u_{x+1}}\right)$$

Therefore this becomes by substitution

$$\Delta \tan^{-1} u_x = \tan^{-1} \frac{1}{a_x}$$

and integrating

$$\tan^{-1} u_x = C + \Sigma \tan^{-1} \frac{1}{a_x}$$

or taking the tangent of each member

$$u_x = \tan \left\{ C + \Sigma \tan^{-1} \frac{1}{a_x} \right\},$$

The preceding is a particular case of this, a_x being there an absolute constant.

SECTION V.

Exercises in the Integration of Equations of Mixed Differences.

It will be necessary in the following cases to adhere to the notation of partial differences we have before employed (See Appendix. Art. 357, 364.) viz.

$$\frac{d}{dx} u_{x,y} = \frac{d u_{x,y}}{dx}; \quad \frac{d}{dy} u_{x,y} = \frac{d u_{x,y}}{dy},$$

and so on for the higher orders of the differential coefficients, thus

$$\left(\frac{d}{dx}\right)^m \left(\frac{d}{dy}\right)^n u_{x,y} = \frac{d^{m+n} u_{x,y}}{dx^m . dy^n}.$$

The same mode of referring the symbols of operation to their proper independent variables may also be conveniently extended to the characteristic Δ, as follows :

$$\frac{\Delta}{\Delta x} u_{x,y} = u_{x+1,y} - u_{x,y}; \quad \frac{\Delta}{\Delta y} u_{x,y} = u_{x,y+1} - u_{x,y},$$

$$\left(\frac{\Delta}{\Delta x}\right)^m \left(\frac{\Delta}{\Delta y}\right)^n u_{x,y} =$$

$$\left(\frac{\Delta}{\Delta y}\right)^n u_{x+m,y} - \frac{m}{1}\left(\frac{\Delta}{\Delta y}\right)^n u_{x+m-1,y} + \&c.$$

These two different modes of varying the independent quantities x, y, may occur together, as in the expression

$$\frac{\Delta}{\Delta x}\,\frac{d}{dy} u_{x,y} = \frac{d\,u_{x+1,y}}{d.y} - \frac{d\,u_{x,y}}{dy}.$$

and others of the like sort, and it is manifestly a matter of indifference in what order the operations denoted by Δ and d are performed. Equations of mixed differences determine the form of a function by assigning a relation between these derivatives. In Appendix. Art. 387. we have considered mixed differential equations with one independent variable. We shall here give a few instances where more than one are involved.

(1). To integrate the equation

$$u_{x+1,y} = \frac{d}{dy} u_{x,y}$$

This gives $$u_{x,y} = \left(\frac{d}{dy}\right)^x \phi(y),$$

$\phi(y)$ being an arbitrary function of y. The reason is evident; for if we take $u_{0,y} = \phi(y)$, we have $u_{1,y} = \frac{d}{dy}\phi(y)$, whence we derive $u_{2,y} = \left(\frac{d}{dy}\right)^2 \phi(y)$, and so on.

(2). Suppose $\frac{\Delta}{\Delta x} u_{x,y} = a\,.\,\frac{d}{dy} u_{x,y}$, Assume $$u_{x,y}=a^x\,.\,e^{-\frac{y}{a}}\,.\,v_{x,y},$$

and

$$\frac{\Delta}{\Delta x}u_{x,y}=e^{-\frac{y}{a}}\{a^{x+1}v_{x+1,y}-a^x v_{x,y}\}$$

$$=a^x\,.\,e^{-\frac{y}{a}}\{a\,v_{x+1,y}-v_{x,y}\}$$

also,

$$a\,.\frac{d}{dy}u_{x,y}=a^x\,.\,e^{-\frac{y}{a}}\,.\left\{a\frac{d}{dy}v_{x,y}-v_{x,y}\right\}.$$

The equation then becomes by substitution and reduction

$$v_{x+1,y}=\frac{d}{dy}v_{x,y}$$

which has already been integrated, and thus we get

$$u_{x,y}=a^x\,.\,e^{-\frac{y}{a}}\,.\left(\frac{d}{dy}\right)^x\phi(y)$$

ϕ denoting an arbitrary function.

(3). Given $u_{x+1,y}=a\,.\left(\frac{d}{dy}\right)^n u_{x,y},$

Assume $u_{x,y}=a^x\,.\,v_{x,y}$ and we find for the integral

$$u_{x,y}=a^x\,.\left(\frac{d}{dy}\right)^{nx}\phi(y)$$

(4). $$u_{x+2,y}-a\left(\frac{d}{dy}\right)u_{x+1,y}+b\left(\frac{d}{dy}\right)^2 u_{x,y}=0,$$

Assume, $u_{x,y}=p^x\,.\,v_{x,y}$, and we get by substitution and division of the whole by p^x,

$$p^2\,v_{x+2,y}-a\,p\,.\frac{d}{dy}v_{x+1,y}+b\,.\left(\frac{d}{dy}\right)^2 v_{x,y}=0.$$

In this equation the sum of the indices $x+2$, $x+1$, x, below the v in each term, and the corresponding exponents 0, 1, 2, of the symbols of differentiation $\frac{d}{dy}$ is the same: if then we suppose

$$v_{x,y} = \left(\frac{d}{dy}\right)^x \phi(y)$$

we shall have

$$v_{x+2,y} = \frac{d}{dy} v_{x+1,y} = \left(\frac{d}{dy}\right)^2 v_{x,y} = \left(\frac{d}{dy}\right)^{x+2} \phi(y),$$

and the whole equation is divisible by this function, leaving

$$p^2 - ap + b = 0,$$

to determine p. Let α and β be its roots, then since the proposed equation is of the first degree, and either $\alpha^x . \left(\frac{d}{dy}\right)^x \phi_1(y)$ or $\beta^x . \left(\frac{d}{dy}\right)^x \phi_2(y)$ separately satisfy it, their sum is the general expression for $u_{x,y}$,

$$u_{x,y} = \alpha^x . \left(\frac{d}{dy}\right)^x \phi_1(y) + \beta^x . \left(\frac{d}{dy}\right)^x \phi_2(y)$$

$\phi_1(y)$ and $\phi_2(y)$ being two arbitrary functions of y.

(5). $$u_{x+n,y} - a\frac{d}{dy} u_{x+n-1,y} + b\left(\frac{d}{dy}\right)^2 u_{x+n-2,y} - \dots \dots \pm k . \left(\frac{d}{dy}\right)^n u_{x,y} = 0.$$

Let α, β, γ, &c. be the roots of

$$u^n - a u^{n-1} + b u^{n-2} - \dots \pm k = 0,$$

then

$$u_{x,y} = \alpha^x . \left(\frac{d}{dy}\right)^x \phi_1(y) + \beta^x . \left(\frac{d}{dy}\right)^x \phi_2(y) + \text{\&c.}$$

$\phi_1(y), \phi_2(y), \dots \phi_n(y)$ denoting n arbitrary functions of y.

But here a remark of considerable importance offers itself. It will immediately be observed that the process by which the above two equations are integrated is entirely independent of the nature of the operation denoted by $\frac{d}{dy}$. It might have been any other, and thus we might by the same process integrate

$$u_{x+n,y} - a\frac{\Delta}{\Delta y}u_{x+n-1,y} + b\left(\frac{\Delta}{\Delta y}\right)^2 u_{x+n-2,y} \dots$$

$$\dots \pm k\left(\frac{\Delta}{\Delta y}\right)^n u_{x,y} = 0,$$

or yet more generally

$$u_{x+n,y} - a\nabla u_{x+n-1,y} + b\nabla^2 u_{x+n-2,y} - \dots$$

$$\dots \pm k\nabla^n u_{x,y} = 0,$$

where $\nabla u_{x,y}$ denotes any linear combination of the differences or differential coefficients of $u_{x,y}$ relative to y, of whatever order we please, nor does its generality stop here.

(6). To integrate the equation

$$\left(\frac{\Delta}{\Delta x}\right)^2 u_{x,y} - a\frac{\Delta}{\Delta x}\frac{d}{dy}u_{x,y} + b\left(\frac{d}{dy}\right)^2 u_{x,y} = 0,$$

Assume $u_{x,y} = \alpha^x . e^{-\frac{y}{a}} . v_{x,y}$,

and we get by substitution, and division by $\alpha^x . e^{-\frac{a}{y}}$,

$$0 = \left\{\alpha^2 v_{x+2,y} - a\alpha . \frac{d}{dy}v_{x+1,y} + b . \left(\frac{d}{dy}\right)^2 v_{x,y}\right\}$$

$$-\frac{2}{a}\left\{\alpha^2 v_{x+1,y} - a\alpha . \tfrac{1}{2}\left(v_{x+1,y} + \frac{d}{dy}v_{x,y}\right) + b . \frac{d}{dy}v_{x,y}\right\}$$

$$+\frac{1}{a^2}\left\{\alpha^2 - a\alpha + b\right\} v_{x,y}.$$

Let us for a moment suppose $v_{x,y} = \left(\frac{d}{dy}\right)^x \phi(y)$, then it is evident by the preceding problems that this equation will become

$$0 = (\alpha^2 - a\alpha + b)\, v_{x+2,y} - \frac{2}{\alpha}.(\alpha^2 - a\alpha + b)\, v_{x+1,y}$$
$$+ \frac{1}{\alpha^2}(\alpha^2 - a\alpha + b)\, v_{x,y}:$$

the factor $\alpha^2 - a\alpha + b$ being found in each term, if this be made to vanish the whole is satisfied, so that provided α be assumed a root of the equation

$$\alpha^2 - a\alpha + b = 0,$$

any error we may have made in our value of $v_{x,y}$ is corrected, and calling α and β its two roots we see that

$$\alpha^x . e^{-\frac{\alpha}{y}} . \left(\frac{d}{dy}\right)^x \phi_1(y) \text{ and } \beta^x . e^{-\frac{\beta}{y}} . \left(\frac{d}{dy}\right)^x \phi_2(y),$$

each satisfy the proposed equation, so that

$$u_{x,y} = \alpha^x . e^{-\frac{\alpha}{y}} \left(\frac{d}{dy}\right)^x \phi_1(y) + \beta^x . e^{-\frac{\beta}{y}} \left(\frac{d}{dy}\right)^x \phi_2(y).$$

(7). $$\left(\frac{\Delta}{\Delta x}\right)^n u_{x,y} - a . \left(\frac{\Delta}{\Delta x}\right)^{n-1} \frac{d}{dy} u_{x,y} +$$
$$+ b . \left(\frac{\Delta}{\Delta x}\right)^{n-2} \left(\frac{d}{dy}\right)^2 u_{x,y} -$$
$$\ldots . \pm k . \left(\frac{d}{dy}\right)^n u_{x,y} = 0.$$

A process similarly conducted will be found to lead to the following result:

$$u_{x,y} = \alpha^x . e^{-\frac{y}{\alpha}} . \left(\frac{d}{dy}\right)^x \phi_1(y) + \beta^x . e^{-\frac{y}{\beta}} . \left(\frac{d}{dy}\right)^x \phi_2(y) + \&c.$$

Where α, β, γ, &c. are the n roots of

$$u^n - a u^{n-1} + \ldots \pm k = 0,$$

and $\phi_1(y), \ldots \phi_n(y)$ are as many arbitrary functions of y.

(8). $u_{x,y,z} = a\left(\frac{d}{a y}\right)^{\alpha} u_{x-1,y,z} + b\left(\frac{d}{d z}\right)^{\beta} u_{x-2,y,z},$

$$u_{x,y,z} = a^x \cdot \left(\frac{d}{a y}\right)^{\alpha x} \phi(y) + b^{\frac{x}{2}} \cdot \left(\frac{d}{a z}\right)^{\frac{\beta x}{2}} \psi(z)$$

$\phi(y)$ and $\psi(z)$ denoting arbitrary functions of y and z.

SECTION VI.

Exercises in the Summation of Series by the Integration of their general Terms.

As the integration of any function leads directly to the sum of the series of which it is the general term (Appendix. Art 389.) the following examples may be looked upon as exercises in that part of the inverse calculus of differences which relates to the integration of explicit functions. To sum then the following series,

(1). $1.2.4 + 2.3.5 + 3.4.6 + \&c.;$

S_x denoting the sum to x terms, we shall find

$$S_x = \frac{x(x+1)(x+2)(3x+13)}{12}.$$

(2). $1^4 + 3^4 + 5^4 + \ldots (2x-1)^4 = S_x$

$$S_x = \frac{x(4x^2-1)(12x^2-7)}{15}.$$

(3). $S_x = 1 + 2p + 3p^2 + 4p^3 + \&c.$

$$S_x = \frac{1 - p^x(1 + x - xp)}{(1-p)^2}.$$

Sum to infinity (when $p < 1$) $= S = \dfrac{1}{(1-p)^2}$.

(4). $\dfrac{10}{1.2.3.4} + \dfrac{14}{2.3.4.5} + \dfrac{18}{3.4.5.6} + \&c.$

$$S_x = \frac{2}{3} - \frac{2}{(x+1)(x+3)}; \quad S = \frac{2}{3}.$$

(5). $\dfrac{1}{\sqrt{2}(1+\sqrt{2})} + \dfrac{1}{(1+\sqrt{2})(2+\sqrt{2})}$

$$+ \frac{1}{(2+\sqrt{2})(3+\sqrt{2})} + \&c.$$

$$S_x = \frac{x}{\sqrt{2}(x+\sqrt{2})}; \quad S = \frac{1}{\sqrt{2}}.$$

(6). $1.3^2 + 3.5^2 + 5.7^2 + \&c.$

$$S_x = \frac{x(6x^3 + 16x^2 + 9x - 4)}{3}.$$

(7). $2.5.8 + 4.8.14 + 8.14.26 + 16.26.50 + \&c.$

$$S_x = \frac{36.2^{3x} + 84.2^{2x} + 56.2^x - 176}{7}.$$

See Appendix. Art. 374.

(8). $\dfrac{3}{5.7} - \dfrac{9}{7.29} + \dfrac{27}{29.79} - \dfrac{81}{79.245} + \&c.$

$$S_x = \frac{1}{20} - \frac{1}{4\{2 + (-3)^x\}}; \quad S = \frac{1}{20}.$$

(9). $\dfrac{1}{2.5.8.11}+\dfrac{3.19}{5.8.11.14}+\dfrac{5.19^2}{8.11.14.17}+\&c.$

$$S_x=\frac{19^x}{27\,(3x+2)(3x+5)(3x+8)}-\frac{1}{2100}.$$ See Ex. 11.

(10). $\dfrac{2}{1.3.5}-\dfrac{5.11}{3.5.7}+\dfrac{8.11^2}{5.7.9}-\&c.$

$$S_x=\frac{1}{24}-\frac{(-11)^x}{8\,(2x+1)(2x+3)}.$$ See Ex. 11.

(11). To sum the series

$$\frac{p}{u_0\,u_1\,u_2\ldots u_{m-1}}+\frac{(p+q)\cdot s}{u_1\,u_2\ldots u_m}+\frac{(p+2q)\,s^2}{u_2\,u_3\ldots u_{m+1}}+\&c.$$

whenever it can be done, and to determine the condition which must be satisfied to render it practicable, (u_0, u_1, u_2, &c. forming an arithmetical progression).

The $(x+1)^{\text{th}}$ term is

$$\frac{(p+q\,x)\,s^x}{u_x\,u_{x+1}\ldots\ldots u_{x+m-1}}.$$

Now the difference of a function

$$\frac{A\cdot s^x}{u_x\ldots\ldots u_{x+m-2}}$$

is easily found to be

$$\frac{A\,(s\,u_x-u_{x+m-1})\,s^x}{u_x\ldots\ldots u_{x+m-1}}.$$

For u_x substitute $a+h\,x$ and the numerator becomes

$$(A\,\{(s-1)\,a-(m-1)\,h\}-A\,(s-1)\,h\,x)\,s^x,$$

which compared with $(p+q\,x)\,s^x$, that of the function in question gives

$$A=\frac{q}{(s-1)\,h},$$

and thus we get for the sum of the series

$$S_x = \frac{q}{(s-1)h}\left\{\frac{s^x}{u_x \ldots\ldots u_{x+m-2}} - \frac{1}{u_0 \ldots\ldots u_{m-2}}\right\},$$

and for the equation of condition

$$\frac{p}{q} = \frac{a}{h} - \frac{m-1}{s-1},$$

also when $s < 1$

$$S = \frac{q}{(1-s)\,h\,.\,u_0 \ldots\ldots u_{m-2}}.$$

The two series immediately preceding this example are particular cases of this.

(12). $\frac{1^2 . 4}{2 . 3} + \frac{2^2 . 4^2}{3 . 4} + \frac{3^2 . 4^3}{4 . 5} + \&c.$

$$S_x = \frac{2}{3} + \frac{4}{3} . \frac{x-1}{x+2} . 4^x.$$

(13). $\frac{1^2 . 9}{3 . 5} + \frac{2^2 . 9^2}{5 . 7} + \frac{3^2 . 9^3}{7 . 9} + \&c.$

$$S_x = \frac{3}{32}\left\{1 + \frac{2x-1}{2x+3} . 3^{2x+1}\right\}.$$

(14). To determine in what cases the function

$$\frac{(p + qx + rx^2)\,s^x}{u_x \ldots\ldots u_{x+m-1}}$$

is integrable, and in those cases to perform the integration.

Assume for the integral

$$\frac{(A + Bx)\,s^x}{u_x \ldots\ldots u_{x+m-2}} \qquad (a.)$$

and by comparing the difference of this with the proposed, we shall find

$$A = \frac{q}{(s-1)h} - \left(\frac{a}{h} + \frac{s-m+1}{s-1}\right) \cdot \frac{r}{(s-1)h}$$

$$B = \frac{r}{(s-1)h},$$

and for the equation of condition

$$p - q \cdot \left(\frac{a}{h}\right) + r\left(\frac{a}{h}\right)^2 =$$

$$= \frac{m-1}{s-1}\left\{ -q + 2r\left(\frac{a}{h}\right) + \frac{s-m+1}{s-1} r \right\}; \qquad (b).$$

Whenever this holds good, the proposed function is directly integrable the integral being as above (*a*).

If we resolve the equation of condition (*b*) with respect to *s*, two values of *s* (real or imaginary) will be found which render the proposed function integrable. This may be yet farther extended, and by a process of the same nature the following theorem may be proved.

(15). THEOREM. *It is always practicable to assign such values of* s, *real or imaginary, being the roots of an equation of the* n[th], *degree that the function*

$$\frac{(\alpha + \beta x + \gamma x^2 + \ldots\ldots \nu x^n) \cdot s^x}{u_x u_{x+1} \ldots\ldots\ldots u_{x+m-1}}$$

shall be directly integrable, $\alpha, \beta, \gamma, \ldots \nu$, *being any given quantities, and* u_x *being of the form* $a + h x$.

(16). To integrate the function

$$\frac{x^2\left(1 + \frac{h}{a}\right)^{2x}}{(hx + a)(hx + a + h)},$$

or to sum the series

$$\frac{1^2}{(a+h)(a+2h)} \cdot \left(\frac{a+h}{a}\right)^2 +$$

$$+ \frac{2^2}{(a+2h)(a+3h)} \cdot \left(\frac{a+h}{a}\right)^4 + \&c.$$

This function satifies the equation of condition (*b*) of (14), and we shall therefore find for the value of the integral, or the sum of the series to $x-1$ terms

$$C+\frac{a^2}{h^3(2a+h)}\cdot\frac{h(x-1)-a}{hx+a}\left(\frac{a+h}{a}\right)^{2x},$$

or, determining the constant, and writing $x+1$ for x,

$$S_x=\frac{a(a+h)}{h^3(2a+h)}\left\{1+\frac{hx-a}{h(x+1)+a}\left(\frac{a+h}{a}\right)^{2x+1}\right\}.$$

This comprehends as a particular case the two series summed in (12) and (13). As series of this sort are not without their interest, especially when considered in an extended point of view, a few more cases are subjoined by way of farther exercise.

(17). $\frac{2}{1.3}\times\frac{1}{3}+\frac{3}{3.5}\times\frac{1}{3^2}+\frac{4}{5.7}\times\frac{1}{3^3}+$ &c.

$$S_x=\frac{1}{4}\left\{1-\frac{1}{(2x+1).3^x}\right\};\quad S=\frac{1}{4}.$$

(18). $\frac{3}{1.2}\times\frac{1}{2}+\frac{4}{2.3}\times\frac{1}{2^2}+$ &c.

$$S_x=1-\frac{1}{(x+1).2^x},\qquad S=1.$$

(19). $\frac{5}{1.2.3}\times\frac{1}{2}+\frac{6}{2.3.4}\times\frac{1}{2^2}+\frac{7}{3.4.5}\times\frac{1}{2^3}+$ &c.

$$S_x=\frac{1}{2}-\frac{1}{(x+1)(x+2).2^x};\quad S=\frac{1}{2}.$$

(20). $1+2+3+\ldots.x=\frac{x(x+1)}{2}$.

(21). $1^2+2^2+3^2+\ldots.x^2=\frac{x(x+1)(x+2)}{3}-\frac{x(x+1)}{2}$.

(22). $1^3 + 2^3 + 3^3 + \ldots\ldots x^3$

$$= \frac{x(x+1)(x+2)(x+3)}{4} - \frac{x(x+1)(x+2)}{1} + \frac{x(x+1)}{2}$$

$$= \left(\frac{x(x+1)}{2}\right)^2.$$

(23). $1^n + 2^n + 3^n + \ldots\ldots x^n = S_x$

$$S_x = (-1)^{n+1}\left\{\Delta o^n . \frac{x(x+1)}{1.2} - \Delta^2 o^n . \frac{x(x+1)(x+2)}{1.2.3} + \ldots \right.$$

$$\left. \ldots\ldots \pm \Delta^n o^n . \frac{x(x+1)\ldots\ldots(x+n)}{1.2.\ldots\ldots(n+1)}.\right\}$$

See Equation (a) of Art. 26. Sect. 2. where $h=1$, $u_x = x+1$.

The same sums may also be exhibited as follows;

(24). $1 + 2 + \ldots\ldots x = \frac{x(x+1)}{1.2}.$

(25). $1^2 + 2^2 + \ldots\ldots x^2 = \frac{1}{8}\left\{\frac{2x+1}{1} + \right.$

$$\left. + \frac{(2x-1)(2x+1)(2x+3)}{3}\right\}.$$

(26). $1^3 + 2^3 + \ldots\ldots x^3 = \frac{(x-1)x(x+1)(x+2)}{4} +$

$$+ \frac{x(x+1)}{2}.$$

(27). $1^4 + 2^4 + \ldots\ldots x^4 = \frac{1}{32}\left\{\frac{2x+1}{1} + \right.$

$$+ 10 . \frac{(2x-1)(2x+1)(2x+3)}{3} +$$

$$\left. + \frac{(2x-3)(2x-1)(2x+1)(2x+3)(2x+5)}{5}\right\}$$

(28). $1^5 + \dots x^5 = \frac{(x-2)(x-1)\dots(x+3)}{6} +$

$$+ 5 . \frac{(x-1)\dots(x+2)}{4} + \frac{x(x+1)}{2}.$$

(29). In general; $1^{2n+1} + 2^{2n+1} + \dots x^{2n+1} =$

$$= A_0 . \frac{x(x+1)}{2} + A_1 . \frac{(x-1)\dots(x+2)}{4} +$$

$$+ A_2 . \frac{(x-2)\dots(x+3)}{6} + \&c.$$

where the coefficients are those given in (Art. 4. Sect. 2.), and

$$1^{2n} + 2^{2n} + \dots x^{2n} = \frac{1}{2^{2n+1}}$$

$$\left\{ \begin{array}{c} A_0 . \frac{2x+1}{1} + A_1 . \frac{(2x-1)(2x+1)(2x+3)}{3} + \&c. \\ - A_0 + 1^2 . A_1 - 1^2 . 3^2 . A_2 + \&c. \end{array} \right\}$$

the coefficients here being those determined in (Art. 6. Sect. 2.)

(30). $1 + 3 + 5 + \dots (2x-1) = x^2$.

(31). $1^2 + 3^2 + 5^2 + \dots (2x-1)^2 =$

$$= \frac{(2x-1) . 2x . (2x+1)}{6}.$$

(32). $1^3 + 3^3 + 5^3 + \dots (2x-1)^3 = 2x^4 - x^2$.

(33). The general expression including all these is

$$1^n + 3^n + 5^n + \dots (2x-1)^n =$$

$$= (-1)^n . \left\{ \frac{\Delta o^{n+1}}{1} . 2^{n-1} . \frac{(2x+1)-1}{1} - \frac{\Delta^2 o^{n+1}}{1 . 2} . 2^{n-2} . \frac{(2x+1)(2x+3) - 1 . 3}{2} + \&c. \right\}$$

(34). $(a+h)^n + (a+2h)^n + (a+3h)^n + \ldots (a+xh)^n =$

$$= u_1^n + u_2^n + \ldots\ldots u_x^n$$

where $u_x = a + hx$;

$$S_x = (-1)^{n+1}\left\{\frac{\Delta 0^n}{1}\,.\,h^{n-2}\,.\,\frac{u_x u_{x+1} - u_0 u_1}{2} - \frac{\Delta^2 0^n}{1\,.\,2} h^{n-3}\,.\,\frac{u_x u_{x+1} u_{x+2} - u_0 u_1 u_2}{3} + \&c.\right\}$$

Expressions for the sums of the powers of the natural numbers were first given by Wallis in his *Arithmetica infinitorum* for the purpose of applying Cavalerius's method of indivisibles to the quadrature of curvilinear spaces whose ordinates are rational integral functions of their abscissæ. Their theory was treated in a more general way by John Bernouilli, and after him by Euler, to whom we are indebted for the general theorem for the expression of Σu_x in a series, whose numerical coefficients (from their identity with those found by John Bernouilli in the case of Σx^n) he called by the name of that Geometer. The expressions given in the above and some following examples for these sums are different we believe from any yet noticed, and seem to be the simplest their nature admits.

(35). $\cos\theta + \cos 3\theta + \cos 5\theta + \ldots \cos(2x-1)\theta =$

$$= \frac{\sin x\theta}{\sin\theta}\,.\cos x\theta. \quad \text{(Append. Art. 373.)}$$

(36). $\sin\theta + \sin 3\theta + \sin 5\theta + \ldots \sin(2x-1)\theta =$

$$= \frac{\sin x\theta}{\sin\theta}\,.\sin x\theta.$$

(37). $\cos\theta + \cos(\theta+h) + \cos(\theta+2h) + \ldots$

$$\cos\{\theta+(x-1).h\} = \cos\left\{\theta + \frac{x-1}{2}h\right\}.\frac{\sin\left(\frac{x}{2}h\right)}{\sin\left(\frac{1}{2}h\right)}.$$

(38). $\sin\theta + \sin(\theta + h) + \sin(\theta + 2h) + \ldots.$

$$\sin\{\theta + (x-1).h\} = \sin\left\{\theta + \frac{x-1}{2}h\right\}.\frac{\sin\left(\frac{x}{2}h\right)}{\sin\left(\frac{1}{2}h\right)}.$$

(39). $1.(\cos\theta)^2 + 2(\cos 2\theta)^2 + 3(\cos 3\theta)^2 + \ldots.$

$$\ldots.x(\cos x\theta)^2 = \frac{x(x+1)}{4} +$$

$$\frac{x}{4.\sin\theta}.\sin(2x+1)\theta - \frac{1}{4}\left(\frac{\sin x\theta}{\sin\theta}\right)^2.$$

(40). $\sin\theta.\cos\psi + \sin 2\theta..\cos 3\psi + \ldots\ldots$

$$\sin x\phi.\cos(2x-1)\psi =$$

$$= \sin\left\{\frac{\theta}{2} + x\left(\frac{\theta}{2} + \psi\right)\right\}.\frac{\sin x\left(\frac{\theta}{2} + \psi\right)}{2\sin\left(\frac{\theta}{2} + \psi\right)} +$$

$$+ \sin\left\{\frac{\theta}{2} + x\left(\frac{\theta}{2} - \psi\right)\right\}.\frac{\sin x\left(\frac{\theta}{2} - \psi\right)}{2\sin\left(\frac{\theta}{2} - \psi\right)}.$$

(41). $\frac{1}{\cos\theta.\cos 2\theta} + \frac{1}{\cos 2\theta.\cos 3\theta} +$

$$\frac{1}{\cos 3\theta.\cos 4\theta} + \&c.$$

$$S =_x \frac{\tan(x+1)\theta - \tan\theta}{\sin\theta};$$ See (Art. 4. Sect 1.)

(42). $\frac{1}{\sin\theta.\cos 2\theta} - \frac{1}{\cos 2\theta.\sin 3\theta} +$

$$+ \frac{1}{\sin 3\theta.\cos 4\theta} - \&c.$$

$$S_x = \frac{1}{\cos\theta \, . \, \sin 2(x+1)\theta}\{(-1)^{x+1} - \cos 2(x+1)\theta\} - \frac{1}{\sin\theta}.$$

(43). $$\frac{1}{\sin\theta \, . \, \sin 2\theta} + \frac{1}{\sin 2\theta \, . \, \sin 3\theta} + \frac{1}{\sin 3\theta \, . \, \sin 4\theta} + \&c.$$

$$S_x = \frac{-\text{cotan}\,(x+1)\theta + \text{cotang}\,\theta}{\sin\theta}.$$

(44). $$\frac{1}{\cos\theta \, . \, \sin 2\theta} - \frac{1}{\sin 2\theta \, . \, \cos 3\theta} + \frac{1}{\cos 3\theta \, . \, \sin 4\theta} - \&c.$$

$$S_x = \frac{(-1)^{x+1} + \cos 2(x+1)\theta}{\cos\theta \, . \, \sin 2(x+1)\theta} + \frac{\tan\theta}{\cos\theta}.$$

(45). $$\frac{1}{\sin\theta \, . \, \sin 3\theta} - \frac{1}{\sin 2\theta \, . \, \sin 4\theta} + \frac{1}{\sin 3\theta \, . \, \sin 5\theta} - \&c.$$

$$S_x = \frac{1}{2 \, . \cos\theta}\left\{\frac{1}{\sin\theta \, . \, \sin 2\theta} + \frac{(-1)^{x+1}}{\sin(x+1)\theta \, . \, \sin(x+2)\theta}\right\}.$$

See (Art. 20. Sect. 1.)

(46). $$\frac{1}{\cos\theta \, . \, \cos 3\theta} - \frac{1}{\cos 2\theta \, . \, \cos 4\theta} + \frac{1}{\cos 3\theta \, . \, \cos 5\theta} - \&c.$$

$$S_x = \frac{1}{2 \, . \cos\theta}\left\{\frac{1}{\cos\theta \, . \, \cos 2\theta} + \frac{(-1)^{x+1}}{\cos(x+1)\theta \, . \cos(x+2)\theta}\right\}.$$

See Art. 22. Sect. 1.

(47). $\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{4}\tan\frac{\theta}{4} + \&c.$

See Art. 14. Sect. 1.

$$S_x = \frac{1}{2^{x-1}}\cot\frac{\theta}{2^{x-1}} - 2\,.\cot 2\,\theta,$$

$$S = \frac{1}{\theta} - 2\,.\cot 2\theta.$$ See Lacroix, Translation, Art. 57.

(48). $(\tan\theta)^2 + \frac{1}{2}\left(\tan\frac{\theta}{2}\right)^2 + \left(\frac{1}{4}\tan\frac{\theta}{4}\right)^2 + \&c.$

$$S_x = \frac{8}{3}\left(1 - \frac{1}{2^{2x}}\right) + \frac{4}{(\tan 2\,\theta)^2} - \frac{1}{\left(2^{x-1}\,.\tan\frac{\theta}{2^{x-1}}\right)^2}.$$

$$S = \frac{8}{3} + \frac{4}{(\tan 2\,\theta)^2} - \frac{1}{\theta^2},$$

See Art. 15. Sect. 1. Also Lacroix. Translation, Art. 57.

(49). $\tan\theta\,.\,(\sec\theta)^2 + \left(\frac{1}{2}\tan\frac{\theta}{2}\right)\left(\frac{1}{2}\sec\frac{\theta}{2}\right)^2 +$

$$+\left(\frac{1}{4}\tan\frac{\theta}{4}\right)\left(\frac{1}{4}\sec\frac{\theta}{4}\right)^2 + \&c.$$

$$S_x = \frac{\cos\frac{\theta}{2^{x-1}}}{\left(2^{x-1}\,.\sin\frac{\theta}{2^{x-1}}\right)^3} - 8\,.\frac{\cos 2\,\theta}{(\sin 2\,\theta)^3}.$$

$$S = \frac{1}{\theta^3} - 8\,.\frac{\cos 2\,\theta}{(\cos 2\,\theta)^3};$$ See Art. 16. Sect. 1. and Translation, Art. 57.

(50). $\frac{a-1}{a+1} + 2\,.\frac{a^2-1}{a^2+1} + 4\,.\frac{a^4-1}{a^4+1} + \&c.$

$$S_x = 2^x \cdot \frac{a^{2^x}+1}{a^{2^x}-1} - \frac{a+1}{a-1}; \qquad \text{(Art. 7. Sect. 1.)}$$

(51). $\dfrac{a}{a^2-1} + \dfrac{a^2}{a^4-1} + \dfrac{a^4}{a^8-1} + \&c.$

$$S_x = \tfrac{1}{2}\left\{\frac{a+1}{a-1} - \frac{a^{2^x}+1}{a^{2^x}-1}\right\}; \qquad \text{(Art. 6. Sect. 1.)}$$

(52). $\dfrac{1}{\sin\theta} + \dfrac{1}{\sin 2\theta} + \dfrac{1}{\sin 4\theta} + \dfrac{1}{\sin 8\theta} + \&c.$

$$S_x = \cot\frac{\theta}{2} - \cot 2^{x-1}\theta; \qquad \text{(Art. 18. Sect. 1.)}$$

(53). $\dfrac{1}{\left(2\,.\cos\frac{\theta}{2}\right)^2} + \dfrac{1}{\left(4\,.\cos\frac{\theta}{4}\right)^2} + \dfrac{1}{\left(8\,.\cos\frac{\theta}{8}\right)^2} + \&c.$

$$S_x = \frac{1}{(\sin\theta)^2} - \frac{1}{\left(2^x\,.\sin\frac{\theta}{2^x}\right)^2}$$

$$S = \frac{1}{(\sin\theta)^2} - \frac{1}{\theta^2}; \quad \text{(Art. 17. Sect. 1.)}$$

(54). $\sin\theta\left(\sin\frac{\theta}{2}\right)^2 + 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{4}\right)^2 +$

$$+ 4\sin\frac{\theta}{4}\left(\sin\frac{\theta}{8}\right)^2 + \&c.$$

$$S_x = \frac{1}{4}\left(2^x\,.\sin\frac{\theta}{2^{x-1}} - \sin 2\theta\right)$$

$$S = \frac{\theta}{2} - \frac{\sin 2\theta}{4}. \quad \text{(See Art. 8. Sect. 8.)}$$

(55). $(\sin \theta)^4 + 4 \left(\sin \frac{\theta}{2}\right)^4 + 4^2 . \left(\sin \frac{\theta}{4}\right)^4 + \&c.$

$$S_x = \left(2^{x-1} \sin \frac{\theta}{2^{x-1}}\right)^2 - \left(\frac{\sin 2\theta}{2}\right)^2,$$

$$S = \theta^2 - \left(\frac{\sin 2\theta}{2}\right)^2; \quad \text{(Art. 9. Sect. 1.}$$

(56). $\tan \theta \left(\tan \frac{\theta}{2}\right)^2 + 2 \tan \frac{\theta}{2} \left(\tan \frac{\theta}{4}\right)^2 +$

$$+ 4 \tan \frac{\theta}{4} \left(\tan \frac{\theta}{8}\right)^2 + \&c.$$

$$S_x = \tan \theta - 2^x . \tan \frac{\theta}{2^x},$$

$$S = \tan \theta - \theta. \quad \text{(Art. 11. Sect. 1.)}$$

(57). $\left(\frac{1}{\cos \theta}\right)^2 + \left(\frac{2}{\cos 2\theta}\right)^2 + \left(\frac{4}{\cos 4\theta}\right) +$

$$+ \left(\frac{8}{\cos 8\theta}\right)^2 + \&c.$$

$$S_x = \left(\frac{2^x}{\sin 2^x \theta}\right)^2 - \frac{1}{\sin \theta^2}.$$

(58). To assign the value of the continued product,

$$P_x = \tan \theta (\tan 2\theta)^{\frac{1}{2}} (\tan 4\theta)^{\frac{1}{4}} \ldots . (\tan 2^x \theta)^{\frac{1}{2^x}}.$$

If we sum the series

$$\log \tan \theta + \tfrac{1}{2} \log \tan 2\theta + \tfrac{1}{4} \log \tan 4\theta + \&c.$$

to $x+1$ terms by the help of (Art. 19. Sect. 1.) and then transform the logarithmic equation into an equation of factors by the well known property

$$a . \log A + b . \log B + \&c. = \log (A^a . B^b . \&c.)$$

we shall find for the value of P_x

$$P_x = \frac{4 . \sin \theta^2}{(2 \sin 2^x + {}^1 \theta)^{\frac{1}{2^x}}},$$

and if P be the product to infinity, $P = 4 . \sin \theta^2$.

(59). To sum the series

$$\tan^{-1} \frac{1}{1 + 1 + 1^2} + \tan^{-1} \frac{1}{1 + 2 + 2^2}$$

$$+ \tan^{-1} \frac{1}{1 + 3 + 3^2} + \&c.$$

or, as it would stand in the ordinary notation,

$$\text{arc}\left(\tan = \frac{1}{1+1+1^2}\right) + \text{arc}\left(\tan = \frac{1}{1+2+2^2}\right) + \&c.$$

to x terms and to infinity

$$S_x = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{x+1}\right); \quad S = \frac{\pi}{4}; \quad \text{(Art. 25. Sect. 1.)}$$

(60). $\tan^{-1} \frac{1}{2 . 1^2} + \tan^{-1} \frac{1}{2 . 2^2} + \tan^{-1} \frac{1}{2 . 3^2} + \&c.$

$$S_x = \frac{\pi}{4} - \tan^{-1} \frac{1}{2x + 1}; \quad S = \frac{\pi}{4};$$

Deducible from 26. Sect. 1. by taking $h + x\theta = 2x + 1$.

(61). $\tan^{-1} \frac{1}{7} + 2 . \tan^{-1} \frac{1}{38} + 2^2 . \tan^{-1} \frac{1}{268} + \&c.$

the progression of the denominators being

$$4 + 3, \quad 4 . 8 + 3 . 2, \quad 4 . 8^2 + 3 . 2^2, \ \&c.$$

$$S_x = 2^x . \tan^{-1} \frac{1}{2^x} - \frac{\pi}{4}; \quad S = 1 - \frac{\pi}{4}; \quad \text{(See 29. Sect. 1.)}$$

(62). $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{27} + \ldots\ldots$

$$\ldots\ldots + \tan^{-1}\frac{2}{3x+5x^2},$$

$$S_x = \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{2}{5x+4}, \quad S = \tan^{-1}\frac{1}{2}.$$

(63). To determine in what cases the function

$$\tan^{-1}\frac{1}{p+qx+rx^2}$$

is immediately integrable, and in such cases to sum the series

$$\tan^{-1}\frac{1}{p+q+r} + \tan^{-1}\frac{1}{p+q.2+r.2^2} +$$

$$+ \tan^{-1}\frac{1}{p+q.3+r.3^2} + \&c.$$

Let the function in question be compared with the second member of the equation in (27. Sect. 1.) and we find

$$a = \frac{1}{\sqrt{(r)}}, \quad b = 0, \quad A = -\frac{q-r}{2\sqrt{(r)}}, \quad B = -\sqrt{(r)},$$

besides which there remains an equation of condition to be satisfied, viz.

$$q^2 - r^2 = 4(pr - 1); \ldots\ldots (a)$$

Whenever this equation then holds good, we have

$$\Sigma \tan^{-1}\frac{1}{p+qx+rx^2} = C - \tan^{-1}\left\{\frac{2}{q-r+2rx}\right\}.$$

The series 59, 60, 62. are all particular cases of this, and their general terms will be found to satisfy the equation (*a*). Other examples are the following.

(64). Let the general or x^{th} term of a series be

$$u_x = \tan^{-1} \frac{1}{10\,x^2 - 24\,x + 12},$$

$$S_x = \tan^{-1}(7) + \tan^{-1}(10\,x - 7); \quad S = \frac{\pi}{2} + \tan^{-1}(7).$$

(65). Let $u_x = \tan^{-1} \dfrac{1}{34\,x^2 - 8\,x - 8}$,

then

$$S_x = \tan^{-1}(34\,x + 13) - \tan^{-1}(13); \quad S = \tan^{-1} \frac{1}{13}.$$

(66). Let $u_x = \tan^{-1} \dfrac{1}{74\,x^2 - 12\,x - 18}$,

then

$$S_x = \tan^{-1}(74\,x + 31) - \tan^{-1}(31); \quad S = \tan^{-1} \frac{1}{31}.$$

(67). Let $u_x = \tan^{-1} \dfrac{1}{26\,x^2 - 16\,x - 4}$,

$$S_x = \tan^{-1}(26\,x + 5) - \tan^{-1}(5); \quad S = \tan^{-1} \frac{1}{5}.$$

The series (47, 48, 49, 50, 53.) are due to Mr. Wallace, who gave them under a somewhat different form, among a variety of similar ones in a paper communicated to the Royal Society of Edinburgh in 1808, as formulæ of approximation to the arc of a circle (when continued to infinity) to which purpose their rapid convergence, even in the most unfavourable cases, well adapts them. In fact we have by (47)

$$\frac{1}{\theta} = (2 \,.\cot 2\,\theta + \tan \theta) + \tfrac{1}{2} \tan \tfrac{1}{2}\,\theta + \&c.$$

in which if we notice that $2 \cot 2\theta + \tan \theta = \frac{1}{\tan \theta}$ we have the reciprocal of the arc, expressed in the form delivered in the paper alluded to. The series (50) when continued backwards by writing $-x$ for x gives

$$\frac{1}{2}\cdot\frac{a^{\frac{1}{2}}-1}{a^{\frac{1}{2}}+1}+\frac{1}{4}\cdot\frac{a^{\frac{1}{4}}-1}{a^{\frac{1}{4}}+1}+\ldots\ldots\frac{1}{2^x}\cdot\frac{a^{2^{-x}}-1}{a^{2^{-x}}+1}=$$

$$=\frac{a+1}{a-1}-\frac{a^{2^{-x}}+1}{2^x\,(a^{2^{-x}}-1)},$$

and this sum, when the series is continued to infinity will be found to reduce itself to

$$\frac{a+1}{a-1}-\frac{2}{\log a}.$$

This expression is accordingly given by Mr. Wallace in the same paper, as affording means of computing the logarithm of an insulated number (a high prime for instance), or at least its reciprocal, at once. It is true the operations are laborious on account of the multiplied extractions of roots and decimal divisions they require, but they are not on that account less valuable. Regarded in the light of elegant formulæ in the inverse method of differences, these series assume a higher rank in the scale of analytical estimation, in proportion to the difficulty of that field of research, and the little reason we have to hope for any farther progress in it. For this reason, I have added the series (52, 54, 55, 56, 57.) which are of a similar nature, but have not been noticed by him. Of these (54, 55, and 56.) afford in like manner, formulæ of approximation to the arc of a circle, viz.

$$\theta = \frac{1}{2}\sin 2\theta +$$

$$+ 2\left\{\sin\theta\,.\left(\sin\frac{\theta}{2}\right)^2 + 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{4}\right)^2 + \&c.\right\}$$

$$\theta^2 = \left(\frac{\sin 2\,\theta}{2}\right)^2 +$$

$$+ \left\{(\sin \theta)^4 + 4\left(\sin \frac{\theta}{2}\right)^4 + 4^2\left(\sin \frac{\theta}{4}\right)^4 + \&c.\right\}; \qquad (a).$$

$$\theta = \tan \theta - \left\{\tan \theta \left(\tan \frac{\theta}{2}\right)^2 + 2 \tan \frac{\theta}{2}\left(\tan \frac{\theta}{4}\right)^2 + \&c.\right\}$$

These all converge with the same degree of rapidity after a few of the first terms, viz. nearly according to the powers of $\frac{1}{4}$, but for actual computation, the formula (*a*) far surpasses the rest in convenience. They differ from Mr. Wallace's in giving the immediate values of the arc and its square instead of their reciprocals.

The continued product (58) is due to Mr. Babbage: the summation of the series of reciprocal sines (52) may be obtained from it, by taking the logarithmic differential relative to θ and *vice versâ*, the latter may be derived from the former by integration: The method, however, in which we have here presented them has the advantange of exhibiting the principle on which all transformations of the same kind ultimately depend.

The series (59) and (60) are noticed by Euler in the Comm. Acad. Petropol. ix. 1737. p. 234., as well as by Spence in his Logarithmic Transcendents. By the former they are given as particular instances of a general formula of great neatness, at which he arrives by a kind of tentative method, but which may be obtained very shortly thus:

$$\text{Since } \Delta \tan^{-1} u_x = \tan^{-1}\left(\frac{\Delta u_x}{1 + u_x u_{x+1}}\right); \text{ 28. Sect. 1.}$$

therefore we have

$$\Sigma \tan^{-1} \frac{\Delta u_x}{1 + u_x u_{x+1}} = C + \tan^{-1} u_x.$$

Thus we get for the sum of the series

$$\tan^{-1}\left\{\frac{u_1 - u_0}{1 + u_0 u_1}\right\} + \tan^{-1}\left\{\frac{u_2 - u_1}{1 + u_1 u_2}\right\} + \ldots\ldots$$

$$\ldots\tan^{-1}\left\{\frac{u_x - u_{x-1}}{1 + u_{x-1} u_x}\right\},$$

the following expression

$$\tan^{-1} u_x - \tan^{-1} u_0.$$

Which is in fact Euler's formula, for u_x being a function of x of a form perfectly arbitrary, its particular values u_0, u_1, u_2, ...u_x to any extent we please, may be looked upon as so many arbitrary and independent constants, and may be represented by separate letters, a, b, c, &c., which done, a very trifling reduction will give the formula in question. These series of inverse tangents in which the numerators of the fractions under the characteristic $\tan^{-1}$ are unity, and the denominators integer numbers (as 59, 60, 61, 62, 64, 65, 66, 67.) are extremely remarkable on account of the facilities they afford for extending the integer evaluation of the function $\tan^{-1}(x)$ or as Spence denotes it ${}^1C(x)$. (64.....7) have not, I believe, been noticed, nor has (61) which is not included in Euler's general formula, but may be derived without difficulty by a method similar to that by which that formula was originally obtained, from the following equation

$$\tan^{-1}\frac{1}{n} + \tan^{-1}\frac{1}{4n^3+3n} = 2\tan^{-1}\frac{1}{2n}; \qquad (a).$$

which the reader will have no difficulty in verifying and which is analogous to a theorem of Borda for calculating the logarithm of a number, by means of three preceding logarithms and a series. The same remark applies to this class of series as to the rest: they are properly and naturally examples of the application of the inverse method of differences, however they may have been originally obtained, and it may not be amiss to shew how any equation such as (a)

expressing a relation between three values of a particular function may become the origin of a similar series.

(68). To sum the series

$$\tan^{-1}\left(\frac{1}{4\,n^3+3\,.\,n}\right) + 2\,.\tan^{-1}\left(\frac{1}{32\,n^3+6\,n}\right) +$$
$$+\ 4\,.\tan^{-1}\left(\frac{1}{256\,n^3+12\,n}\right) \text{ \&c.}$$

The $(x+1)^{\text{th}}$ term is

$$2^x\,.\tan^{-1}\left\{\frac{1}{4\,(2^x\,n)^3+3\,.\,2^x\,n}\right\} = u_{x+1}.$$

Now in (a), let $2^x n$ be written for n, and the whole multiplied by 2^x, and we get

$$2^x\,.\tan^{-1}\left\{\frac{1}{4\,(2^x\,n)^3+3\,.\,2^x\,n}\right\} =$$

$$= 2^{x+1}\tan^{-1}\frac{1}{2^{x+1}\,n} - 2^x\tan^{-1}\frac{1}{2^x\,n}.$$

The second member of this equation is evidently the complete difference of $2^x\tan^{-1}\dfrac{1}{2^x\,n}$, so that integrating both members,

$$\Sigma u_{x+1} = C + 2^x\,.\tan^{-1}\frac{1}{2^x\,n}$$

and if the constant be determined as usual we find

$$S_x = 2^x\,.\tan^{-1}\frac{1}{2^x\,n} - \tan^{-1}\frac{1}{n},\quad S = \frac{1}{n} - \tan^{-1}\frac{1}{n}.$$

(69). Let $f^0(n) = n$, $f(n) = 4\,n^3 + 3\,n$,

$$f^2(n) = 4f(n)^3 + 3\,.f(n), \text{ \&c.}$$

Required the sum of the series

$$\tan^{-1}\frac{1}{2f^0(n)} - \tan^{-1}\frac{1}{2f(n)} + \tan^{-1}\frac{1}{2f^2(n)} - \&c.$$

The $(x+1)^{\text{th}}$ term, or u_{x+1} is

$$u_{x+1} = (-1)^x . \tan^{-1}\frac{1}{2f^x(n)}.$$

Now if in the equation (a) we substitute $f^x(n)$ for n and multiply the whole by $(-1)^x$ we obtain

$$(-1)^x . \tan^{-1}\frac{1}{f^{x+1}(n)} + (-1)^x . \tan^{-1}\frac{1}{f^x(n)} =$$

$$= 2(-1)^x . \tan^{-1}\frac{1}{2f^x(n)}.$$

The first member of this equation is the exact difference of

$$(-1)^{x+1} . \tan^{-1}\frac{1}{f^x(n)}$$

because the two terms of which it consists are the successive values of this function due to the variation of x, with contrary signs, and that without any regard to the form of the function $f^x(n)$ considered as a function of n; the second member is equal to $2\,u_{x+1}$; Hence

$$2\,u_{x+1} = \Delta . (-1)^{x+1} \tan^{-1}\frac{1}{f^x(n)}$$

$$\Sigma\, u_{x+1} = \frac{(-1)^{x+1}}{2} \tan^{-1}\frac{1}{f^x(n)} + C,$$

whence we obtain for the sum of the series

$$S_x = \tfrac{1}{2}\left\{\tan^{-1}\frac{1}{n} + (-1)^{x+1}\tan^{-1}\frac{1}{f^x(n)}\right\};$$

$$S = \tfrac{1}{2}\tan^{-1}\frac{1}{n},$$

for, whatever be the value of n, the quantities n, $f(n)$, $f^2(n)$, $f^3(n)$, &c. form an increasing series which diverges with extreme rapidity. Thus, if $n=1$, these successive values are

$$1, 7, 1393, 10812186007, \&c.$$

If n then be any integer number, $\frac{1}{f^3(n)}$ and the following values may even be altogether disregarded, in a numerical point of view. If we have detained the reader too long on this point, its close connexion with the quadrature of the circle, will induce him to pardon the digression. We will now resume the subject.

(70). The series 1, 5, 17, 53, 161, 485, &c. is a recurring one.—What are—its scale of relation,—its general term, and its sum to x terms?

The scale is..............

The general term is........

The sum is $3^x - x - 1$.

(71). Which of the two series

1, 0, 3, 2, 6, 11, 23, 49, 223, &c.

1, 0, 3, 2, 5, 10, 24, 51, 247, &c.

is a recurring one, and what is its scale of relation?

(72). To shew that

$$\sin\theta,\ \sin 2\theta,\ \sin 3\theta, \ldots\ldots \sin x\theta,$$

$$\text{and } \cos\theta,\ \cos 2\theta,\ \cos 3\theta, \ldots\ldots \cos x\theta,$$

form two recurring series, and to find their scales of relation.

The reader will remember in order to prove this, that the character of a recurring series consists in the possibility of expressing any term by one or more of the preceding terms, multiplied by invariable quantities.

SECTION VII.

Problems and Theorems relating to the developement of exponential Functions, and the properties of the numbers comprised in the form $\Delta^m o^n$.

THE equation

$$\Delta^n u_x = (e^{\frac{dx}{d}} - 1)^n u_x$$

discovered by Lagrange (See Appendix, Art. 358.) and the yet more general theorem of Arbogast demonstrated in the following article, render it desirable to possess some general formula, to facilitate the developement of these and similar expressions. We have already seen some of the uses to which the numbers comprised in the form $\Delta^m o^n$ are applicable in the theory of series. In what follows we shall lay before the reader a connected view of their properties, which bear directly upon the point in question, and afford an easy and general solution of the difficulty. But their application is by no means confined to this, and before we quit this subject we shall point out their use in one or two other instances where they may be introduced with advantage.

(1). PROBLEM. To develope $f(e^t)$ any function whatever of e^t, in a series of the powers of t, or to determine A_0, A_1, A_2, &c, in the following equation,

$$f(e^t) = A_0 + A_1 t + A_2 t^2 + \ldots\ldots A_x t^x + \&c.$$

Let $f(1)$, $f'(1)$, $f''(1)$, &c. denote the values assumed by $f(x)$, $\frac{df(x)}{dx}$, $\frac{d^2 f(x)}{dx^2}$, &c. the several differential coefficients or derived functions of $f(x)$ when x becomes unity. Then by Taylor's theorem we shall have (h being any quantity)

$$f(1+h)=f(1)+\frac{f'(1)}{1}h+\frac{f''(1)}{1.2}h^2+\&c.$$

Since this is true whatever be the value of h, suppose it equal to e^t-1, and the above equation becomes

$$f(e^t)=f(1)+\frac{f'(1)}{1}(e^t-1)+\frac{f''(1)}{1.2}(e^t-1)^2+\&c.$$

The coefficient of t^x therefore in the first member $f(e^t)$ is equal to the sum of its several coefficients in the terms of the second member. Now, the coefficient of t^x in $f(1)$ is $f(1)\times 0^x$ being $f(1)$ when $x=o$ and zero in all other cases.

In $\frac{f'(1)}{1}(e^t-1)$ it is $\frac{f'(1)}{1}\cdot\frac{1^x-0^x}{1.2\ldots.x}=$

$$=\frac{f'(1)}{1}\cdot\frac{\Delta o^x}{1\ldots\ldots x}.$$

In $\frac{f''(1)}{1.2}(e^t-1)^2$, or $\frac{f''(1)}{1.2}(e^{2t}-2e^t+1)$ it is

$$\frac{f''(1)}{1.2}\times\frac{2^x-2.1^x+0^x}{1\ldots\ldots x}=\frac{f''(1)}{1.2}\cdot\frac{\Delta^2 o^x}{1\ldots\ldots x}$$

and so on; (as is evident if we consider that the developement of e^{nt} in general is

$$1+\frac{n}{1}t+\frac{n^2}{1.2}t^2+\ldots.\frac{n^x}{1\ldots.x}t^x+\&c.)$$

Let these be collected together, and we find for the value of A_x or the coefficient in $f(e^t)$

$$A_x=\frac{1}{1.2\ldots.x}\left\{f(1).o^x+\frac{f'(1)}{1}\Delta o^x+\frac{f''(1)}{1.2}\Delta^2 o^x+\&c.\right\}$$

Now let the symbols of operation be separated from those of quantity, and we get

$$A_x = \frac{1}{1.2\ldots x}\left\{f(1) + \frac{f'(1)}{1}\Delta + \frac{f''(1)}{1.2}\Delta^2 + \&c.\right\}o^x$$
$$= \frac{f(1+\Delta)\,o^x}{1.2\ldots x}.$$

$f(1+\Delta)\,o^x$ being understood (as in all similar cases) to have no other meaning than its developement, of which it is a mere abbreviated expression, each power of Δ being understood to be separately affixed *as if* by multiplication to the 0^x which follows. Hence this general

THEOREM.

$$f(e^t) = f(1) + \frac{t}{1}f(1+\Delta)\,o + \frac{t^2}{1.2}f(1+\Delta)\,o^2 + \&c.$$

which will be found to comprise all the properties of the numbers $\Delta^m o^n$ we shall have occasion to employ.

(2). Corollary. Hence, if by any of the usual methods the developement of $f(e^t)$ be obtained, or the value of A_x assigned, that of $f(1+\Delta)\,o^x$ may be obtained in functions of x and *vice versâ*, for we have

$$A_x = \frac{f(1+\Delta)\,o^x}{1\ldots x}$$

and,

$$f(1+\Delta)\,o^x = 1.2\ldots x.\,A_x$$

For example, we shall have,

(3). $(1+\Delta)\,o^x = 1^x, \quad (1+\Delta)^2 o^x = 2^x, \ldots (1+\Delta)^n o^x = n^x,$

whatever value we assign to n, whether positive, negative, integral, fractional, or even imaginary. For,

let $f(1+\Delta) = (1+\Delta)^n$, then $f(e^t) = e^{nt}$, or since

$$e^{nt} = 1 + \frac{n}{1}t + \ldots \frac{n^x}{1\ldots x}t^x + \&c.$$

we have

$$1 \, . \, 2 \ldots . x \, . \, A_x = n^x$$

the value of $f(1 + \Delta) o^x$ to be found.

Suppose for instance $n = -1$, and we have

$$\frac{1}{1 + \Delta} o^x = (-1)^x.$$

Now the first member of this (by the definition) has no other meaning than

$$\{ 1 - \Delta + \Delta^2 - \&c. \} o^x$$

or,
$$o^x - \Delta o^x + \Delta^2 o^x - \&c.$$

But in this *(as in all other such series) we may omit all the terms after* $\Delta^x o^x$, *they being each separately zero, by the property of these numbers,* (Appendix, Art. 350.) so that we get,

$$o^x - \Delta o^x + \Delta^2 o^x \ldots . \pm \Delta^x o^x = (-1)^x,$$

or, merely reversing the order of writing it,

$$\Delta^x o^x - \Delta^{x-1} o^x + \ldots . \pm \Delta o^x \mp o^x = 1,$$

whatever be the value of x. It may not be amiss to verify this result by a numerical example, suppose for instance $x = 5$, and taking the values of $\Delta^5 o^5$, $\Delta^4 o^5$, &c. from the Table (Ex. 33. Sect. 1.) we have

$$120 - 240 + 150 - 30 + 1 = 1.$$

(4.) Again, we may prove in like manner, that

$$\{ \log (1 + \Delta) \}^n o^x = 0,$$

unless $n = x$, in which case its value is $1 \, . \, 2 \ldots . n$. For since $(\log e^t)^n = t^n$, the coefficient of t^x in the developement of this function (regarded as a function of e^t) is zero unless $x = n$, in which case we have A_x or $A_n = 1$, and $1 \, . \, 2 \ldots . n \, A_n = 1 \, . \, 2 \ldots . n$.

(5). Thus taking $n=1$, we have

$$\frac{\Delta\, 0^x}{1} - \frac{\Delta^2 0^x}{2} + \ldots\ldots \pm \frac{\Delta^x 0^x}{x} = 0,$$

for every value of x greater than unity, which may in like manner be verified by numerical substitution.

(6). $\{\log(1+\Delta)\}^n . f(\Delta)\, 0^x = x(x-1) \ldots\ldots$
$\ldots\ldots(x-n+1) f(\Delta)\, 0^{x-n}$.

The coefficient of t^x in the developement of $t^n . f(\epsilon^t - 1)$ is evidently the same with that of t^{x-n} in that of $f(\epsilon^t - 1)$. The former coefficient is

$$\frac{\{\log(1+\Delta)\}^n f(\Delta)\, 0^x}{1 . 2 \ldots x},$$

and the latter, $\dfrac{f(\Delta)\, 0^{x-n}}{1 . 2 \ldots (x-n)}$,

which being equated, the proposition is apparent.

(7). $\Delta^{-n} 0^x = \dfrac{1 . 2 \ldots x}{1 . 2 \ldots (x+n)} \left\{\dfrac{\log(1+\Delta)}{\Delta}\right\}^n 0^{x+n}$.

An immediate consequence of the foregoing, changing only x into $x+n$, and making $f(\Delta) = \Delta^{-n}$. This equation enables us to continue the series,

$$\Delta^0 0^x, \ \Delta\, 0^x, \ \Delta^2 0^x, \ \&c.,$$

backwards, to any extent, according to one uniform law, though it must rather be regarded as a definition of $\Delta^{-n} 0^x$ than in any other light, since the value of that expression (or its equivalent $\Sigma^n 0^x$) is not fixed by assigning only the superior limit (0) of the integral.

(8). **Prop.** To shew that whatever be the value of n,

$$f\{(1+\Delta)^n\}\, 0^x = n^x . f(1+\Delta)\, 0^x.$$

Take the identical equation

$$f\ \{ (e^t)^n \} = f(e^{nt}).$$

The coefficient of t^x in the first member of this equation (regarded as a function of e^t) is by (1. Sect. 7.)

$$\frac{f\ \{ 1 + \Delta)^n \}\ o^x}{1 \,.\, 2 \ldots . x}$$

but, in the second it is evidently equal to $n^x \times$ into that of the same power of t in $f(e^t)$, or to

$$n^x \,.\, \frac{f(1 + \Delta)\, o^x}{1 \,.\, 2 \ldots . x}.$$

Equating these the transformation in question results, which is often of great use in eluding very troublesome developements. Thus for instance.

(9).
$$\{ 1 + \sqrt{(1 + \Delta)} \}^m o^x = \frac{1}{2^x} \,.\, (2 + \Delta)^m o^x,$$

$$\{ (1 + \Delta)^n - 1 \}^m o^x = n^x \,.\, \Delta^m o^x,$$

$$\{ 1 + (1 + \Delta)^n \}^m o^x = n^x \,.\, (2 + \Delta)^m o^x.$$

(10). To prove the following very general properties of the numbers comprised in the form $\Delta^n o^m$,

$$\left\{ f(1 + \Delta) + f\left(\frac{1}{1 + \Delta}\right) \right\} o^{2x-1} = o ; \ldots . (a),$$

$$\left\{ f(1 + \Delta) - f\left(\frac{1}{1 + \Delta}\right) \right\} o^{2x} = 0 ; \ldots . (b).$$

whatever be the form of the function denoted by f.

Suppose

$$f(e^t) = A_0 + A_1 t + A_2 t^2 + \text{\&c.}$$

then will $\quad f(e^{-t}) = A_0 - A_1 t + A_2 t^2 - \text{\&c.}$

Hence, it is evident that their sum $f(e^t)+f(e^{-t})$ contains no odd powers of x, and their difference $f(e^t)-f(e^{-t})$ no even ones. The coefficients therefore of t^{2r-1} in the developement of the former, and of t^{2r} in that of the latter expression are respectively zero. Now these expressions, put under the form $f(e^t) \pm f\left(\frac{1}{e^t}\right)$ and regarded as functions of e^t give, by applying the general theorem (Ex. 1. Sect. 7.), for the aforesaid coefficients, the first members respectively of the equations (a) and (b), whence the truth of the proposition is apparent. It may also be derived from (8) by making $n=-1$. Many particular cases of these theorems assume a very remarkable form, thus:

(11). If we take $f(1+\Delta) = \frac{1}{1+(1+\Delta)}$,

we have by (b)

$$\frac{\Delta}{2+\Delta}\, o^{2r} = 0,$$

or

$$\frac{\Delta\, o^{2r}}{2} - \frac{\Delta^2 o^{2r}}{2^2} + \frac{\Delta^3 o^{2r}}{2^3} \ldots \pm \frac{\Delta^{2r} o^{2r}}{2^{2r}} = 0.$$

(12). If we suppose $f(1+\Delta) = \{1-(1+\Delta)\}^n$, we have $f\left(\frac{1}{1+\Delta}\right) = \left\{1-\frac{1}{1+\Delta}\right\}^n = \left(\frac{\Delta}{1+\Delta}\right)^n$, and the above theorem gives the two equations

$$\left(\frac{\Delta}{1+\Delta}\right)^n o^{2r-1} = (-1)^{n+1}\, \Delta^n o^{2r-1},$$

and

$$\left(\frac{\Delta}{1+\Delta}\right)^n o^{2r} = (-1)^n\, \Delta^n o^{2r},$$

both which may be included in one, by writing it as follows:

$$\left(\frac{\Delta}{1+\Delta}\right)^n 0^x = (-1)^{n+x} \Delta^n 0^x.$$

(13). Let us take a transcendental form of f, and suppose

$$f(1+\Delta) = \int \frac{d\Delta}{\Delta} \log(1+\Delta),$$

then we have it demonstrated (in Note N, p. 683 to Lacroix, Engl. Transl.) that

$$f(1+\Delta) + f\left(\frac{1}{1+\Delta}\right) = \tfrac{1}{2}\log(1+\Delta)^2,$$

but by (4) it appears that $\log(1+\Delta)^2 0^{2x} = 0$, unless $2x = 2$, or $x = 1$, therefore (this case excepted)

$$f\left(\frac{1}{1+\Delta}\right)0^{2x} = -f(1+\Delta)\,0^{2x},$$

which substituted in (b) of (10) gives

$$f(1+\Delta)\,0^{2x} = 0.$$

Now the form of $f(1+\Delta)$ in this instance being the transcendent,

$$\int \frac{d\Delta}{\Delta} \log(1+\Delta) = \frac{\Delta}{1^2} - \frac{\Delta^2}{2^2} + \&c.$$

our equation becomes

$$\frac{\Delta\, 0^{2x}}{1^2} - \frac{\Delta^2\, 0^{2x}}{2^2} + \ldots \pm \frac{\Delta^{2x}\, 0^{2x}}{(2x)^2} = 0,$$

which will be found verified in every case ($x = 1$ excepted) by actual substitution of the numerical values given in the table. To such an extent may the separation of the symbols of operation from those of quantity be carried, without the possibility of error or misconception. What value the first member of the above equation assumes, when the exponent of 0 is odd, will be seen hereafter (22. Sect. 8.)

(14). THEOREM. Let $f_1(\Delta)$ and $f_2(\Delta)$ be any two functions of Δ, then will the following equation hold good

$$\{f_1(\Delta)\cdot f_2(\Delta)\}\,o^x = f_1(\Delta)\cdot f_2(\Delta')\,\{o+o'\}^x,$$

where in the second member, the unaccented Δ is to be referred to the unaccented powers of 0, and the accented to the accented powers.

Let

$$f_1(e^t-1) = A_0 + A_1 t + A_2 t^2 + \&c.$$

$$f_2(e^t-1) = a_0 + a_1 t + a_2 t^2 + \&c.$$

then will

$$f_1(e^t-1)\times f_2(e^t-1) =$$

$$A_0 a_0 + \{A_1 a_0 + A_0 a_1\}\,t + \{A_2 a_0 + A_1 a_1 + A_0 a_2\}\,t^2 + \&c.$$

The coefficient of t^x in the first member of this, by the general theorem in (1. Sect. 7.) is represented by

$$\frac{\{f_1(1+\Delta-1)\times f_2(1+\Delta-1)\}\,o^x}{1\ldots\ldots x} =$$

$$\frac{\{f_1(\Delta)\times f_2(\Delta)\}\,o^x}{1\ldots\ldots x}, \qquad (a).$$

while in the second it is

$$A_x a_0 + A_{x-1} a_1 + \ldots\ldots A_0 a_x, \qquad (b).$$

But, because A_x and a_x are the coefficients of t^x in the respective developements $f_1(e^t-1)$ and $f_2(e^t-1)$ we have by the same general theorem

$$A_x = \frac{f_1(\Delta)\,o^x}{1\ldots\ldots x}, \quad a_x = \frac{f_2(\Delta)\,o^x}{1\ldots\ldots x},$$

hence we find, $A_{x-1} = \dfrac{f_1(\Delta)\,o^{x-1}}{1\ldots\ldots(x-1)}$, &c.

$$a_0 = f_2(\Delta)\,o^0, \quad a_1 = \frac{f_2(\Delta)\,o^1}{1}, \ \&c.$$

and by substitution, the expression (b) becomes

$$\frac{1}{1\ldots.x}\left\{f_1(\Delta)\,o^x\,.f_2(\Delta)\,o^0+\frac{x}{1}f_1(\Delta)\,o^{x-1}\,.f_2(\Delta)\,o^1+\right.$$
$$\left.+\frac{x(x-1)}{1.2}f_1(\Delta)\,o^{x-2}\,.f_2(\Delta)\,o^2+\&c.\ldots\ldots\right\}$$

Now let the symbols of operation be separated from those of quantity, keeping the powers of 0 distinct from each other, by the system of accentuation explained in Appendix, Art. 355, and it becomes

$$\frac{1}{1\ldots.x}f_1(\Delta)f_2(\Delta')\{o+o'\}^x,$$

which compared with the expression (a) renders the proposition evident.

(15). By a process precisely similar, we may prove in general that

$$\{f_1(\Delta)\times f_2(\Delta)\times f_3(\Delta)\times\&c.\}\,o^x=$$
$$f_1(\Delta)\,.f_2(\Delta').\,\&c.\{o+o'+o''+\&c.\}^x.$$

(16). Hence also we may shew that whatever be the value of n,

$$\{(1+\Delta)^n f(\Delta)\}\,o^x=f(\Delta)\{n+o\}^x,$$

$(n+o)^x$ being developed in powers of o as a mere algebraic symbol, and $f(\Delta)$ being then applied to each separate power so produced. For, if in the foregoing proposition (14) we write $(1+\Delta)^n$ for $f_1(\Delta)$ and $f(\Delta)$ for $f_2(\Delta)$ we see that

$$(1+\Delta)^n f(\Delta)\,o^x=$$
$$(1+\Delta)^n o^x\,.f(\Delta)\,o^0+\frac{x}{1}(1+\Delta)^n o^{x-1}\,.f(\Delta)\,o^1+\&c.$$

Now by 3. Sect. 7.) it appears that

$$(1+\Delta)^n o^x=n^x,\quad(1+\Delta)^n o^{x-1}=n^{x-1},\ \&c.$$

so that the second member becomes

$$n^x . f(\Delta) o^0 + \frac{x}{1} n^{x-1} . f(\Delta) o^1 + \&c.$$

which separating the symbols of operation from those of quantity, takes the form

$$f(\Delta) \left\{ n^x o^0 + \frac{x}{1} n^{x-1} o^1 + \frac{x(x-1)}{1 . 2} n^{x-2} o^2 + \&c. \right\}$$

$$= f(\Delta) \{ n + o \}^x.$$

(17). As a particular instance of the application of this let $f(\Delta) = \Delta^m$ and let $n = -1$, and we shall get

$$\Delta^m (-1 + o)^x = \frac{\Delta^m}{1 + \Delta} o^x,$$

or since $\Delta^m (-1 + o)^x = (-1)^x . \Delta^m (1 - o)^x$,

$$\Delta^m (1 - o)^x = (-1)^x . \frac{\Delta^m}{1 + \Delta} o^x$$

$$= (-1)^x . \{ \Delta^m o^x - \Delta^{m+1} o^x + \dots \pm \Delta^x o^x \}.$$

This is the transformation we have already had occasion to employ in (25. Sect. 1.), and that made use of in (27. Sect. 1.) may be derived precisely in the same manner; for if instead of putting $n = -1$, in the theorem in (14), we put $n = -a$, still supposing $f(\Delta) = \Delta^m$, it becomes

$$\Delta^m (-a + o)^x = \frac{\Delta^m}{(1 + \Delta)^a} o^x,$$

whence,

$$\Delta^m (a - o)^x = (-1)^x . \frac{\Delta^m}{(1 + \Delta)^a} o^x.$$

and the use of these transformations in simplifying pretty complicated expressions, and reducing them to a manageable and even elegant form, is in the instances alluded to (and expecially the latter) by no means contemptible. If we make

$m=0$, the terms of $\Delta^m(a-o)^x$ after the first $(\Delta^0 a^x . o^0 = a^x . \Delta^0 o^0 = a^x)$ all vanish, and we have simply

$$a^x = (-1)^x . \frac{1}{(1+\Delta)^a} o^x$$

as we there asserted. Other uses of the transformations in this and the last number will shortly appear.

(18). THEOREM. $f(\Delta)\{(a+o)^x . (b+o)^y . (c+o)^z . \&c.\} =$

$$= f(\Delta) . (1+\Delta')^a . (1+\Delta'')^b . \&c.$$

$$\{(o+o')^x . (o+o'')^y . (o+o''')^z . \&c.\}$$

The Δ's and their powers being referred by the accents over them to the powers of 0, affected with the same number of accents.

To demonstrate it, we have

$$f(\Delta)(a+o)^x . (b+o)^y =$$

$$a^x f(\Delta)(b+o)^y + \frac{x}{1}(a^{x-1} f(\Delta) o (b+o)^y + \&c.$$

$$= a^x . \left\{ b^y f(\Delta) o^0 + \frac{y}{1} b^{y-1} f(\Delta) o^1 + \right.$$

$$\left. \frac{y(y-1)}{1.2} b^{y-2} f(\Delta) o^2 + \&c. \right\}$$

$$+ \frac{x}{1} a^{x-1} \left\{ b^y f(\Delta) o^1 + \frac{y}{1} b^{y-1} f(\Delta) o^2 + \&c. \right\}$$

$$+ \frac{x(x-1)}{1.2} a^{x-2} \times \&c. + \&c.$$

Now we have, $b^y = (1+\Delta)^b o^y$; $b^{y-1} = (1+\Delta)^b o^{y-1}$; &c. by (3. Sect. 7.) and substituting these values in the above expression it becomes

$$a^x \left\{ (1+\Delta)^b o^y . f(\Delta) o^0 + \frac{y}{1} (1+\Delta)^b o^{y-1} . f(\Delta) o^1 + \&c. \right\}$$

$$+\frac{x}{1}a^{x-1}\left\{(1+\Delta)^b o^y . f(\Delta)o^1 + \frac{y}{1}(1+\Delta)^b o^{y-1} f(\Delta) o^2 + \&c.\right\}$$

$$+\frac{x(x-1)}{1.2}.\&c.$$

In this we may now separate the symbols of operation from those of quantity, by employing the system of accentuation, and we shall have for the value of our expression,

$$a^x . f(\Delta).(1+\Delta'')^b \left\{o''^y . o^0 + \frac{y}{1}o''^{y-1}o^1 + \&c.\right\}$$

$$+\frac{x}{1}a^{x-1} . f(\Delta).(1+\Delta'')^b \left\{o''^y . o^1 + \frac{y}{1}o''^{y-1}o^2 + \&c.\right\} + \&c.$$

The series within the brackets have for their abbreviations respectively, $o^0(o+o'')^y$, $o^1(o+o'')^y$, $o^2(o+o'')^y$, &c. and writing these in their places, and at the same time replacing a^x, a^{x-1}, &c. by their values (given by 3. Sect. 7.)

$$a^x = (1+\Delta')^a o'^x;\ a^{x-1} = (1+\Delta')^a o'^{x-1},\ \&c.$$

our formula once more transformed will be

$$f(\Delta).(1+\Delta'')^b \{o^0(o+o'')^y\} . (1+\Delta')^a o'^x +$$

$$+\frac{x}{1} . f(\Delta)(1+\Delta'')^b \{o^1(o+o'')^y\} . (1+\Delta')^a o'^{x-1} + \&c.$$

and again, finally separating the symbols of operation from those of quantity, it will become

$$f.(\Delta)(1+\Delta')^a(1+\Delta'')^b\left\{(o+o'')^y\left(o'^x o^0 + \frac{x}{1}o'^{x-1}o^1 + \&c.\right)\right\}$$

$$= f(\Delta)(1+\Delta')^a(1+\Delta'')^b\{(o+o')^x . (o+o'')^y\},$$

and by a similar train of operations the theorem in question may be proved, to the full extent of its enunciation.

SECTION VIII.

Application of the foregoing Theorems to the developement of particular Functions, the Summation of Series, &c.

(1). To develope $\frac{1}{1+e^t}$ in powers of t.

$$f(e^t) = \frac{1}{1+e^t} = A_0 + A_1 t + A_2 t^2 + \&c.$$

$$f(1+\Delta) o^x = \frac{1}{1+(1+\Delta)} o^x = \frac{1}{2+\Delta} o^x.$$

Therefore, by Ex. 2. Sect. 7.

$$A_x = \frac{1}{1.2\ldots x} \times \frac{1}{2+\Delta} o^x,$$

$$= \frac{1}{1.2\ldots x} \left\{ \frac{o^x}{2} - \frac{\Delta o^x}{2^2} + \ldots \pm \frac{\Delta^x o^x}{2^{x+1}} \right\}.$$

Thus we find $A_0 = 1 . \frac{0^0}{2} = \frac{1}{2}$, $A_1 = \frac{1}{1}\left(0 - \frac{1}{4}\right) = -\frac{1}{4}$,

$A_2 = 0$, and so on, so that

$$\frac{1}{1+e^t} = \frac{1}{2} - \frac{t}{4} + t^2 . \&c.$$

This is the great advantage resulting from the employment of these functions: any series of them such as

$$a . o^x + b \Delta o^x + c \Delta^2 o^x + \&c.$$

however complicated, necessarily breaks off in a limited number $(x+1)$ of terms, and thus enables us to assign in a comparatively simple form, the general terms of an unlimited variety of developements, which would otherwise be scarcely expressible without having recourse to the combinatory analysis, which ought never, in my opinion, to be employed, till every artifice of abbreviation, and every refinement of analysis has been found unavailing.

(2). To devolope $(e^t-1)^n$ in powers of t,

$$f(e^t)=(e^t-1)^n;\quad f(1+\Delta)\,o^x=(1+\Delta-1)^n\,o^x=\Delta^n o^x;$$

$$(e^t-1)^n=\Delta^n o^0+\frac{\Delta^n o^1}{1}\,t+\frac{\Delta^n o^2}{1\,.\,2}\,t^2+\text{\&c.}$$

$$=\frac{\Delta^n o^n}{1\ldots\ldots n}\,t^n+\frac{\Delta^n o^{n+1}}{1\,.2\ldots(n+1)}\,t^{n+1}+\text{\&c.}$$

This elegant expression was originally given by Mr. Ivory (Leybourne's Repository, 1804. Quest. 60.) and afterwards by Dr. Brinkley, Phil. Trans. 1807. i. Both these Geometers arrive at it, however, by a different method from that above pursued. The former mentions it only incidentally, nor does the latter, who pursued the subject much farther, seem to have perceived the *system* of which it, and several more of the truly beautiful theorems he has given in that paper, form a part. To him, however, belongs the merit of introducing the numbers comprised in the form $\Delta^m o^n$ among the *data* of analysis, as objects of ultimate reference. I ought too, to notice that the developement of $\frac{1}{1+e^t}$ in the form above given (1. Sect. 8.) is also to be found for the first time in his paper.

(3). To develope $\frac{t}{e^t-1}$ in powers of t,

$$\frac{t}{e^t-1}=\frac{\log(e^t)}{e^t-1}=f(e^t);$$

$$f(1+\Delta)o^x=\frac{\log(1+\Delta)}{(1+\Delta)-1}\,o^x=\left\{\frac{o^x}{1}-\frac{\Delta o^x}{2}+\ldots\ldots\pm\frac{\Delta^x o^x}{x+1}\right\},$$

and consequently the coefficient of t^x, or

$$A_x = \frac{1}{1\,.\,2\ldots\ldots x}\left\{\frac{0^x}{1} - \frac{\Delta\,0^x}{2} + \frac{\Delta^2\,0^x}{3}\ldots\ldots \pm \frac{\Delta^x\,0^x}{x+1}\right\}.$$

This expression for the coefficient of t^x is given in the Phil. Trans. 1815. in a paper, "On the developement of Exponential Functions."

It has been already shewn (Appendix, Art. 408.) that the odd values of A_x in the developement of $\frac{t}{e^t - 1}$ (A_1 excepted) all vanish, hence we see that the following equation must hold good for every value of x except unity,

$$\frac{\Delta\,0^{2x-1}}{2} - \frac{\Delta^2\,0^{2x-1}}{3} + \ldots\ldots + \frac{\Delta^{2x-1}\,0^{2x-1}}{2\,x} = 0.$$

From the same article, it also appears that the coefficient of t^{2x} in the same developement is

$$(-1)^{x+1}\,.\,\frac{B_{2x-1}}{1\,.\,2\ldots.2\,x},$$

which compared with $\frac{f(1+\Delta)\,0^{2x}}{1\,.\,2\ldots.2\,x}$, gives the following very simple expression for the numbers of Bernouilli,

$$B_{2x-1} = (-1)^{x+1}\,.f(1+\Delta)\,0^{2x}$$

$$= (-1)^{x+1}\,.\,\frac{\log\,(1+\Delta)}{\Delta}\,0^{2x}$$

or, $$B_{2x-1} = (-1)^{x+1}\left\{\frac{0^{2x}}{1} - \frac{\Delta\,0^{2x}}{2}\ldots\ldots \pm \frac{\Delta^{2x}\,0^{2x}}{2\,x+1}\right\}.$$

Thus we may calculate the numerical values of these numbers, with a degree of facility far surpassing that afforded by the expression demonstrated in that article, for instance we have

$$B_1 = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6};\quad B_3 = +\frac{1}{2} - \frac{14}{3} + \frac{36}{4} - \frac{24}{5} = \frac{1}{30},$$

and so on.

(4). To develope $\left(\frac{t}{e^t - 1}\right)^n$ in powers of t.

This function being equal to $\left(\frac{\log(e^t)}{e^t - 1}\right)^n$, the coefficient of t^x will be by (Ex. 1. Sect. 7.)

$$\frac{1}{1.2\ldots.x} \times \left\{\frac{\log(1+\Delta)}{\Delta}\right\}^n o^x.$$

For the developement of this function into a formula adapted to numerical computation, the reader is referred to the paper "*on the Developement of Exponential Functions*" above cited. Or he will find the original function completely developed in a most elegant series, in Dr. Brinkley's paper above noticed. Our object is to remark that since by (Ex. 7. Sect. 7.) we have

$$\frac{1}{1\ldots.(x+n)} \cdot \left\{\frac{\log(1+\Delta)}{\Delta}\right\}^n o^{x+n} = \frac{1}{1\ldots.x} \Delta^{-n} o^x,$$

therefore, the coefficient of t^{x+n} in the proposed function, or, which is the same thing, that of t^x in

$$(e^t - 1)^{-n}$$

is properly represented by $\frac{\Delta^{-n} o^x}{1.2\ldots.x}$, and therefore that the equation

$$(e^t - 1)^n = \frac{\Delta^n o^n}{1\ldots.n} t^n + \frac{\Delta^n o^{n+1}}{1\ldots.(n+1)} t^{n+1} + \&c.$$

proved by Dr. Brinkley to subsist for positive values of n, is now shewn to hold good also for negative.

(5). To develope e^{e^t} in powers of t.

$$f(e^t) = e^{e^t}, \quad f(1+\Delta)o^x = e^{1+\Delta}o^x = e.e^{\Delta}o^x,$$

whence we deduce

$$A_x = \frac{e}{1.2....x} \times e^{\Delta} o^x,$$

$$= \frac{e}{1.2....x}\left\{o^x + \frac{\Delta o^x}{1} + \frac{\Delta^2 o^x}{1.2} + ... \frac{\Delta^x o^x}{1....x}\right\}.$$

Thus $A_0 = e$, $A_1 = e$, $A_2 = \frac{2e}{1.2}$, $A_3 = \frac{5e}{1.2.3}$, &c.

and

$$e^{e^t} = e + e\frac{t}{1} + 2e.\frac{t^2}{1.2} + 5e.\frac{t^3}{1.2.3} + \&c.$$

(6). To develope $\frac{1}{\sqrt{(1+e^t)}}$ and in general $(a + e^t)^n$ in powers of t,

$f(e^t) = (a+e^t)^n$; $f(1+\Delta) = (1+a+\Delta)^n$,

$$A_x = \frac{1}{1.2....x} \times \{(1+a)+\Delta\}^n o^x,$$

$$= \frac{(1+a)^n}{1.2....x}\left\{o^x + \frac{n}{1}.\frac{\Delta o^x}{1+a} + \frac{n(n-1)}{1.2}.\frac{\Delta^2 o^x}{(1+a)^2} + ... \right.$$
$$\left. .... + \frac{n(n-1)....(n-x+1)}{1.2....x}.\frac{\Delta^x o^x}{(1+a)^x},\right\}$$

and in the case proposed where $a=1$ and $n=-\frac{1}{2}$, the value of A_x is

$$\frac{1}{\sqrt{2}.1.2....x}\left\{o^x - \frac{1}{4}\Delta o^x + \frac{1.3}{4.8}\Delta^2 o^x - .... \right.$$
$$\left. .... \pm \frac{1.3....(2x-1)}{4.8....(4x)}\Delta^x o^x,\right\}$$

and in the same manner may any algebraic function of e^t be developed with little trouble.

(7). To extend the above mode of developing $f(\epsilon^t)$ to exponential functions of two or more variables of the form $f(\epsilon^t, \epsilon^{t'}, \epsilon^{t''}, \&c.)$

Let all but t be regarded as so many constants, entering into the composition of the given function, regarded as a function of ϵ^t. Then will the coefficient of any power of t, as t^x be, by what has been proved,

$$\frac{f\{1+\Delta, \epsilon^{t'}, \epsilon^{t''}, \&c.\} o^x}{1 . 2 . . . . x},$$

Into this function t does not enter, but, being a function of $\epsilon^{t'}$, &c. it is itself developable in a series of power of t', t'', &c. Let the coefficient of t'^y in this developement be sought by a similar process. It will be

$$\frac{f\{1+\Delta, 1+\Delta', \epsilon^{t''}, \&c.\} o^x . o'^y}{1 . 2 . . . . x \times 1 . 2 . . . . y},$$

the powers of Δ produced by this second process being kept distinct from those resulting from the first. and applied to their proper power o^y of o, by the accents affixed to them as in (Ex. 14. Sect. 7.). This then will be the coefficient of $t^x . t'^y$, in the developement of the proposed function, regarded as a function only of t and t'. Proceeding in this manner, and denoting by $A_{x, y, z, \&c.}$ the coefficient of the combination $t^x \cdot t'^y . t''^z . \&c.$ in the final result, we have

$$A_{x, y, z, \&c.} = \frac{f\{1+\Delta, 1+\Delta', 1+\Delta'', \&c.\} o^x . o'^y . o''^z . \&c.}{1 . 2 . . . . x \times 1 . 2 . . . . y \times 1 . 2 . . . . z \times \&c.}.$$

(8). To develope $f\{\epsilon^{t+t'+t''+\&c.}\}$ in powers of t, t', &c..

In this case

$$A_{x, y, z, \&c.} = \frac{f\{(+\Delta)(1+\Delta')(1+\Delta'') . \&c.\} o^x . o'^y . o''^z . \&c.}{1 . 2 . . . . x \times 1 . 2 . . . . y \times 1 . 2 . . . . z \times \&c.}.$$

Now let $K(1+\Delta)^n . (1+\Delta')^{n'} . \&c.$ be any term of the developement of $f\{(1+\Delta)(1+\Delta') . \&c.\}$. . Then when this is prefixed to $o^x . o'^y . o''^z$. &c. in the manner denoted by the accents, it becomes

$$K . (1+\Delta)^n o^x . (1+\Delta)^n o^y . (1+\Delta)^n o^z . \&c.$$
$$= K . n^x . n^y . n^z . \&c. \text{ (by 3. Sect. 7.)}$$
$$= K . n^{x+y+z+\&c.} = K (1+\Delta)^n o^{x+y+z+\&c.}.$$

Hence it appears that the same series of terms will result from

$$f \{ (1+\Delta)(1+\Delta') . \&c. \} o^x . o'^y . \&c.$$

as would have arisen from

$$f(1+\Delta) o^{x+y+z+\&c.},$$

and consequently, that

$$A_{x,y,z,\&c.} = \frac{f(1+\Delta) o^{x+y+z+\&c.}}{1.2....x \times 1.2....y \times 1.2....z \times \&c.},$$

Which is also deducible from the theorem for raising a polynomial to any power, combined with that in (Ex. 1. Sect. 7.) and *vice versâ*, the multinomial theorem is directly deducible from this.

Hence too it appears, that the developement of $f(e^{t+t'+\&c.})$ is directly deducible from that of $f(e^t)$; the coefficient of $t^x . t'^y . t''^z$. &c. in the former being equal to that of $t^{x+y+z+\&c.}$ in the latter, multiplied into

$$\frac{1.2....(x+y+z+\&c.)}{1....x \times 1....y \times 1....z + \&c.}.$$

(9). To prove that

$$\frac{1}{2+\Delta} o^{2x-1} = (-1)^x . \frac{2^{2x}-1}{2x} . B_{2x-1},$$

B_{2x-1} being the x^{th} number of Bernouilli.

By (Ex. 1. Sect. 8.) $\frac{1}{2+\Delta} o^{2x-1}$ is the coefficient of

t^{2x-1} in the devolopement of $\frac{1}{1+e^t}$, multiplied by $1 . 2 . 3 \ldots (2x-1)$, that is, to the coefficient of t^{2x} in $\frac{t}{1+e^t}$, multiplied by the same quantity. Again, by Appendix, Art. 408. it appears that this last coefficient is equal to $-(2^{2x}-1) \times$ into the coefficient of the same power in $\frac{t}{e^t-1}$, which coefficient is $(-1)^{x+1} . \frac{B_{2x-1}}{1 . 2 \ldots . 2x}$. Hence we must have

$$\frac{1}{2+\Delta} o^{2x-1} =$$

$$1 . 2 \ldots . (2x-1) \times -(-1)^{x+1} . \frac{2^{2x}-1}{1 . 2 \ldots . 2x} . B_{2x-1}$$

$$= (-1)^x . \frac{2^{2x}-1}{2x} . B_{2x-1}.$$

(10). To sum the series

$$1^n - 2^n + 3^n - 4^n + \&c. \textit{ ad inf.}$$

Since by (Ex. 3. Sect. 7.) we have $(1+\Delta) o^n = 1^n$, $(1+\Delta)^2 o^n = 2^n$, &c. therefore the proposed series becomes by substitution

$$S = (1+\Delta) o^n - (1+\Delta)^2 o^n + (1+\Delta)^3 o^n - \&c.$$

or, separating the symbols of operation from those of quantity

$$S = \{ (1+\Delta) - (1+\Delta)^2 + (1+\Delta)^3 - \&c. \} o^n$$

$$= \frac{1+\Delta}{1+(1+\Delta)} o^n = \left\{ 1 - \frac{1}{2+\Delta} \right\} o^{2n},$$

because this function developed in powers of Δ will necessarily produce the same series as the other. To throw this into a calculable form, let it be actually developed in this manner, and we find

$$S = \frac{o^n}{2} + \frac{\Delta\, o^n}{2^2} - \frac{\Delta^2 o^n}{2^3} + \ldots \pm \frac{\Delta^n o^n}{2^{n+1}}.$$

Thus we have

$$1 - 1 + 1 - 1 + \&c. = \frac{1}{2},$$

$$1 - 2 + 3 - 4 + \&c. = \frac{1}{4},$$

$$1^2 - 2^2 + 3^2 - 4^2 + \&c. = 0,$$

$$1^3 - 2^3 + 3^3 - 4^3 + \&c. = \frac{1}{8},$$

$$1^4 - 2^4 + 3^4 - 4^4 + \&c. = 0;\ \&c.$$

All the even values vanish, and it ought to be so, for the substitution of $-\frac{1}{2\,\{1+(1+\Delta)\}}$ for $f\,(1^t + \Delta)$ in (10. Sect. 7.) equation (*b*), gives

$$\frac{1+\Delta}{1+(1+\Delta)}\, o^{2x} = \frac{o^{2x}}{2} = 0,$$

in every case except when $x = 0$, when it becomes $\frac{1}{2}$. The odd values may be expressed by the numbers of Bernouilli, for writing $2x - 1$ for n, the general expression for the odd values of S becomes

$$-\frac{1}{2+\Delta}\, o^{2x-1} = (-1)^{x+1} . \frac{2^{2x}-1}{2x} B_{2x-1}$$

by (Ex. 9. Sect. 8.): a result exactly agreeing with that deduced by Euler in his *Institutiones Calculi differentialis*, Cap. vii. p. 501. from a principle, it must be confessed, not at all satisfactory.

(11). The series

$$1^n + 2^n + 3^n + \ldots x^n = S_x$$

being proposed, if we treat it in the same way, we get

$$S_x = \{(1+\Delta) + (1+\Delta)^2 + \ldots (1+\Delta)^x\} o^n$$

$$= \frac{(1+\Delta)^{x+1} - (1+\Delta)}{\Delta} o^n$$

which, developed in powers of Δ, gives

$$S_x = x \cdot o^n + \frac{(x+1)x}{1 \cdot 2} \Delta o^n + \frac{(x+1)x(x-1)\Delta^2 o^n}{1 \cdot 2 \cdot 3} + \ldots\ldots$$

$$\ldots\ldots + \frac{(x+1)x \ldots (x-n+1)}{1 \cdot 2 \ldots (n+1)} \Delta^n o^n.$$

This is different in its form, from any expression we have yet given for the sum of this series. It may, however, be obtained by resolving $(x+1)^n$ into preceding, instead of succeeding values, and integrating, as in (Ex. 26. Sect. 2. and Ex. 23. Sect. 6.)

(12). To sum the series of (10) viz.

$$1^n - 2^n + 3^n - \&c.$$

to x terms.

$$\text{Here } S_x = \{(1+\Delta) - (1+\Delta)^2 + (1+\Delta)^3 \ldots\ldots$$

$$\pm (1+\Delta)^x\} o^n$$

$$= \frac{(1+\Delta) - (-1)^x (1+\Delta)^{x+1}}{2+\Delta} o^n.$$

This expression, developed in powers of Δ affords a calculable value of S_x, the number of whose terms can never exceed $n+1$, and consequently the developement need never be carried farther. Thus we have, for instance,

$$1 - 2 + 3 - \ldots \pm x = (-1)^{x+1} \cdot \left(\frac{2x+1}{4}\right) + \frac{1}{4},$$

$$1^2 - 2^2 + 3^2 - \ldots \pm x^2 = (-1)^{x+1} \cdot \frac{x(x+1)}{2},$$

$$1^3 - 2^3 + 3^3 \dots \pm x^3 =$$

$$(-1)^{x+1} \cdot \tfrac{1}{2}\left(x^3 + \frac{3}{2}x^2 - \frac{1}{4}\right) - \frac{1}{8}; \text{ \&c.}$$

(13). To sum the series

$$1^n \cdot t + 2^n \cdot t^2 + 3^n \cdot t^3 + \text{\&c.}$$

to x terms and to infinity. This series treated exactly in the same manner, becomes

$$S_x = \{ t(1+\Delta) + t^2(1+\Delta)^2 + \dots t^x(1+\Delta)^x \} o^n$$

$$= \frac{t^{x+1}(1+\Delta)^{x+1} - t(1+\Delta)}{t(1+\Delta) - 1} o^n.$$

Let S represent the series to infinity, then

$$S = \frac{t(1+\Delta)}{(1-t) - t\Delta} o^n = \left\{ \frac{t}{1-t} + \frac{t\Delta}{(1-t)(1-t-t\Delta)} \right\} o^n$$

$$= \frac{t}{1-t}\left\{ 1 + \frac{\Delta}{1-t} + \frac{t\Delta^2}{(1-t)^2} + \text{\&c.} \right\} o^n,$$

$$= \frac{1}{1-t}\left\{ t \cdot o^n + \left(\frac{t}{1-t}\right)\Delta o^n + \left(\frac{t}{1-t}\right)^2 \Delta^2 o^n + \dots \right.$$

$$\left. \dots \left(\frac{t}{1-t}\right)^n \Delta^n o^n \right\}.$$

Thus we have as particular instances

$$1 \cdot t + 2 \cdot t^2 + 3 \cdot t^3 + \text{\&c.} = \frac{t}{(1-t)^2},$$

$$1^2 \cdot t + 2^2 \cdot t^2 + 3^2 \cdot t^3 + \text{\&c.} = \frac{t}{(1-t)^2} + \frac{2t^2}{(1-t)^3}.$$

The expression for S_x may, in like manner, be easily developed in powers of Δ, and will then assume a calculable form, but we prefer leaving it in its present state. The reader may, if he please, supply this part of the operation.

(14). In general to sum the series

$$A_0 . o^n + A_1 . 1^n + A_2 . 2^n + \&c.$$

having given

$$A_0 + A_1 . t + A_2 . t^2 + \&c. = F(t).$$

The series treated as before gives

$$S = \{ A_0 + A_1 (1 + \Delta) + \&c. \} o^n$$

$$= F(1 + \Delta) o^n$$

$$= F(1) . o^n + \frac{F'(1)}{1} \Delta o^n + \&c.$$

when adapted to actual computation by the developement of $F(1 + \Delta)$ in powers of Δ.

(15). To sum the series

$$S = 1^n + \frac{2^n . h}{1} + \frac{3^n . h^2}{1 . 2} + \frac{4^n . h^3}{1 . 2 . 3} + \&c.$$

This series being the same with

$$o^{n+1} + \frac{1^{n+1}}{1} h + \frac{2^{n+1}}{1 . 2} h^2 + \&c.$$

by substituting e^{ht} for $F(t)$ and $n + 1$ for n in the last problem we find

$$S = e^{h(1+\Delta)} o^{n+1}$$

$$= e^h \left\{ o^{n+1} + \frac{\Delta o^{n+1}}{1} h + \frac{\Delta^2 o^{n+1}}{1 . 2} h^2 + \ldots\ldots \right.$$

$$\left. \ldots . \frac{\Delta^{n+1} o^{n+1}}{1 \ldots . (n+1)} h^{n+1} \right\}.$$

Thus, if we suppose $h = 1$, we find

$$1 + \frac{2}{1} + \frac{3}{1 . 2} + \&c. = 2 e,$$

$$1^2 + \frac{2^2}{1} + \frac{3^2}{1 \,.\, 2} + \&c. = 5\,e,$$

$$1^3 + \frac{2^3}{1} + \frac{3^3}{1 \,.\, 2} + \&c. = 15\,e, \text{ and so on.}$$

(16). $1^n - 3^n + 5^n - \&c. = S.$

Then,

$$S = \frac{1+\Delta}{1+(1+\Delta)^2}\,o^n = \frac{1+\Delta}{2(1+\Delta)+\Delta^2}\,o^n.$$

The developement of this in powers of Δ would be attended with some trouble, but this is not indispensable. It may be reduced to a limited number of *calculable* terms in many different ways, the simplest of which seems to be as follows:

$$S = \left\{\frac{1+\Delta}{2(1+\Delta)} - \frac{(1+\Delta)\,\Delta^2}{4\,(1+\Delta)^2} + \frac{(1+\Delta)\,\Delta^4}{8\,(1+\Delta)^3} - \&c.\right\} o^n.$$

$$= \frac{o^n}{2} - \frac{1}{4}\cdot\frac{\Delta^2}{1+\Delta}\,o^n + \frac{1}{8}\cdot\frac{\Delta^4}{(1+\Delta)^2}\,o^n - \&c.$$

the number of whose terms can never exceed $\frac{n}{2}+1$ and each term of which is easily calculated. In fact, whenever n is an odd number S vanishes, for, if in the equation (a) of (10. Sect. 7.) we suppose

$$f(1+\Delta) = \frac{1+\Delta}{1+(1+\Delta)^2},$$

there will result

$$\frac{1+\Delta}{1+(1+\Delta)^2}\,o^{2x-1} = 0.$$

and when n is even (writing $2\,x$ for it) the value of S becomes

$$\frac{o^{2x}}{2} - \frac{1}{4}\cdot\frac{\Delta^2}{1+\Delta}\,o^{2x} + \ldots \pm \frac{1}{2^{x+1}}\cdot\frac{\Delta^{2x}}{(1+\Delta)^x}\,o^{2x};$$

Thus we find

$$1 - 1 + 1 - 1 + \&c. = \frac{1}{2}$$

$$1 - 3 + 5 - 7 + \&c. = 0,$$

$$1^2 - 3^2 + 5^2 - 7^2 + \&c. = -\frac{1}{2}$$

$$1^3 - 3^3 + 5^3 - 7^3 + \&c. = 0, \&c.$$

See also Note *D*. *Lacroix, Transl.* p. 360.

(17). To complete the theory of these sums we shall subjoin the following example, which will call for the employment of nearly all the transformations above demonstrated, and will thus, by illustrating their use and management, prove the more acceptable as the mode of investigation followed in this and the last section is, if we mistake not, perfectly novel in analysis.

To sum the series

$$\frac{1}{1^{2r+1}} - \frac{1}{3^{2r+1}} + \frac{1}{5^{2r+1}} - \&c. \textit{ ad inf.}$$

and to express its sum by means of the numbers of Bernoulli.

Let C_{2r+1} represent the sum of the series, or

$$C_1 = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \&c.$$

$$C_3 = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \&c.$$

Then Euler has shewn that, θ being any arc, the developement of sec θ will be

$$\sec\theta = \frac{2^2}{\pi} \cdot C_1 + \frac{2^4}{\pi^3} C_3 \cdot \theta^2 + \frac{2^6}{\pi^5} C_5 \cdot \theta^4 + \&c.$$

So that C_{2r+1} is equal to $\left(\frac{\pi}{2}\right)^{2r+1} \times \frac{1}{2}$ the coefficient of θ^{2x} in the developement of sec θ. Call this coefficient A_{2x}, then since

$$\sec \theta = \frac{1}{\cos \theta} = \frac{2}{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}},$$

we have by (1. Sect. 7.)

$$A_{2x} = \frac{1}{1 . 2 . . . . (2x)} \times \frac{2}{(1 + \Delta)^{\sqrt{-1}} + (1 + \Delta)^{-\sqrt{-1}}} o^{2x},$$

but since by (8. Sect. 7.), writing $2x$ for x and $\sqrt{-1}$ for n

$$f \{ (1 + \Delta)^{\sqrt{-1}} \} o^{2x} = (\sqrt{-1})^{2x} . f(1 + \Delta) o^{2x},$$
$$= (-1)^{x} . f(1 + \Delta) o^{2x},$$

we have

$$A_{2x} = \frac{2 . (-1)^{x}}{1 . 2 . . . . (2x)} \times \frac{1}{(1 + \Delta) + (1 + \Delta)^{-1}} o^{2x},$$

which substituted in the value of C_{2x+1} gives

$$C_{2x+1} = \frac{(-1)^{x} . \left(\frac{\pi}{2}\right)^{2x+1}}{1 . 2 . . . . (2x)} . \frac{1 + \Delta}{(1 + \Delta)^{2} + 1} o^{2x}.$$

The latter factor of this expression coincides precisely with that obtained in the last number for the series

$$1^{2x} - 3^{2x} + 5^{2x} - \&c.$$

and its numerical value may readily be computed by the formula there given. Hence this remarkable relation between the two series

$$1 . 2 . 3 . . . . (2x) \times \left\{ \frac{1}{1^{2x+1}} - \frac{1}{3^{2x+1}} + \frac{1}{5^{2x+1}} - \&c. \right\} =$$
$$= (-1)^{x} . \left(\frac{\pi}{2}\right)^{2x+1} \times \{ 1^{2x} - 3^{2x} + 5^{2x} - \&c. \}$$

and as particular instances,

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \&c. = \frac{\pi}{4}$$

$$\frac{1}{1^{3}} - \frac{1}{3^{3}} + \frac{1}{5^{3}} - \&c. = \frac{\pi^{3}}{32}, \&c.$$

The transformation into numbers of Bernouilli may be accomplished as follows.

$$\frac{1+\Delta}{(1+\Delta)^2+1}=\frac{(1+\Delta)^2}{(1+\Delta)\{(1+\Delta)^2+1\}}=$$

$$=\frac{1}{2(1+\Delta)}\left\{1+\frac{(1+\Delta)^2-1}{(1+\Delta)^2+1}\right\}$$

$$=\frac{1}{2(1+\Delta)}+\frac{1}{2}\cdot\frac{(1+\Delta)-(1+\Delta)^{-1}}{(1+\Delta)^2+1}.$$

Now, by (3. Sect. 7.) it appears that $\frac{1}{1+\Delta}o^{2x}=1$, and we therefore have

$$\frac{1+\Delta}{(1+\Delta)^2+1}o^{2x}=\frac{1}{2}+\frac{1}{2}\frac{(1+\Delta)-(1+\Delta)^{-1}}{(1+\Delta)^2+1}o^{2x}.$$

$$=\frac{1}{2}+2^{2x-1}\cdot\frac{(1+\Delta)^{\frac{1}{2}}-(1+\Delta)^{-\frac{1}{2}}}{(1+\Delta)+1}o^{2x},\qquad\text{(by 8. Sect. 7.)}$$

$$=\frac{1}{2}+2^{2x-1}\cdot\frac{(1+\Delta)^{\frac{1}{2}}-(1+\Delta)^{-\frac{1}{2}}}{2+\Delta}o^{2x}.$$

Now, in the theorem (16. Sect. 7.) if we make $f(\Delta)=\frac{1}{2+\Delta}$, and $n=\frac{1}{2}$ and $-\frac{1}{2}$ in succession, it gives

$$\frac{(1+\Delta)^{\frac{1}{2}}}{2+\Delta}o^{2x}=\frac{1}{2+\Delta}\left(o+\frac{1}{2}\right)^{2x};$$

$$\frac{(1+\Delta)^{-\frac{1}{2}}}{2+\Delta}o^{2x}=\frac{1}{2+\Delta}\left(o-\frac{1}{2}\right)^{2x};$$

so that by substitution we get

$$\frac{1+\Delta}{(1+\Delta)^2+1}o^{2x}=\frac{1}{2}+2^{2x-1}\cdot\frac{1}{2+\Delta}\left\{\left(o+\frac{1}{2}\right)^{2x}-\left(o-\frac{1}{2}\right)^{2x}\right\}$$

$$=\frac{1}{2}+\frac{2^{2x-1}}{2+\Delta}\left\{\frac{2x}{1}o^{2x-1}+\frac{2x(2x-1)(2x-2)}{1.2.3\times2^2}o^{2x-3}+\&c.\right\}$$

Again, we have, by (9. Sect. 8.)

$$\frac{1}{2+\Delta} o^{2x-1} = (-1)^x \cdot \frac{2^{2x}-1}{2x} B_{2x-1}$$

whence,

$$\frac{1}{2+\Delta} o^{2x-3} = -(-1)^x \cdot \frac{2^{2x-2}-1}{2x-2} B_{2x-3}; \text{ \&c.}$$

and consequently, by substituting these values, our expression is cleared of the symbols Δ and o, and reduces itself to

$$\frac{1}{2} + (-1)^x \cdot 2^{2x-1} \left\{ \frac{2^{2x}-1}{1} B_{2x-1} \right.$$

$$\left. - \frac{2x(2x-1)}{1 \cdot 2} \cdot \frac{2^{2x-2}-1}{3 \cdot 2^2} B_{2x-3} + \text{\&c.} \right\}$$

and hence we have, finally,

$$C_{2x+1} = \frac{\left(\frac{\pi}{2}\right)^{2x+1}}{1 \cdot 2 \ldots (2x)} \left\{ \frac{(-1)^x}{2} + \right.$$

$$+ 2^{2x-1} \left(\frac{2^{2x}-1}{1} B_{2x-1} - \frac{2x(2x-1)}{1 \cdot 2} \cdot \frac{2^{2x-2}-1}{3 \times 2^2} B_{2x-3} + \right.$$

$$\left. \left. + \ldots \frac{2x \ldots 3}{1 \cdot 2 \ldots (2x-2)} \cdot \frac{2^2-1}{(2x-1) \cdot 2^{2x-2}} B_1 \right) \right\}.$$

The practicability of expressing this function by means of the numbers of Bernouilli was first shewn, and the above expression demonstrated in the Phil. Trans. 1814. The demonstration there given is however very circuitous and rather obscure. The above has the advantage of connecting what was before an insulated result with the general theory of series of this sort.

(18). Given the sum of the series

$$A_0 + A_1 \cdot t + A_2 \cdot t^2 + \text{\&c.} = F(t)$$

or the generating function of A_x, required the sum of

$$A_1 \cdot 1^n \cdot t + A_2(1^n + 2^n) \cdot t^2 + A_3(1^n + 2^n + 3^n)t^3 + \&c.$$

or the generating function of

$$A_x \cdot (1^n + 2^n + \ldots . x^n).$$

By substituting for $1^n, 2^n, \ldots . x^n$, their values

$$(1+\Delta)o^n, \ (1+\Delta)^2 o^n, \ldots . (1+\Delta)^x o^n$$

we find as in (11. Sect. 8.)

$$1^n + 2^n + \ldots . x^n = \frac{(1+\Delta)^{x+1} - (1+\Delta)}{\Delta} o^n,$$

So that our series becomes,

$$A_1 t \cdot \frac{(1+\Delta)^2 - (1+\Delta)}{\Delta} o^n + A_2 t^2 \cdot \frac{(1+\Delta)^3 - (1+\Delta)}{\Delta} o^n + \&c.$$

$$= \frac{1+\Delta}{\Delta} \{ A_1 t(1+\Delta) + A_2 t^2 (1+\Delta)^2 + \&c. \} o^n -$$

$$- \frac{1+\Delta}{\Delta} o^n \cdot \{ A_1 t + A_2 t^2 + \&c. \}$$

$$= \frac{1+\Delta}{\Delta} \{ F \cdot (1+\Delta)t - A_0 \} o^n - \frac{1+\Delta}{\Delta} o^n \{ F(t) - A_0 \}$$

$$= \frac{1+\Delta}{\Delta} \{ F(t + t\Delta) - F(t) \} o^n$$

(19). For example

$$\frac{1^n}{1} + \frac{1^n + 2^n}{1 \cdot 2} + \frac{1^n + 2^n + 3^n}{1 \cdot 2 \cdot 3} + \&c. =$$

$$= e \cdot (1+\Delta) \frac{e^{\Delta} - 1}{\Delta} o^n = e \left\{ 1 + \frac{3}{1 \cdot 2} \Delta + \frac{4}{1 \cdot 2 \cdot 3} \Delta^2 + \&c. \right\} o^n$$

$$= e \left\{ o^n + \frac{3}{1 \cdot 2} \Delta o^n + \frac{4}{1 \cdot 2 \cdot 3} \Delta^2 o^n + \right.$$

$$\left. + \ldots \ldots \frac{n+2}{1 \cdot 2 \ldots . (n+1)} \Delta^n o^n \right\}.$$

(20). $1^n . t + (1^n + 2^n) t^2 + (1^n + 2^n + 3^n) t^3 +$ &c. ad inf.

$$S = \frac{1+\Delta}{\Delta} \left\{ \frac{t}{1-t-t\Delta} - \frac{t}{1-t} \right\} o^n$$

$$= \frac{t^2}{(1-t)^2} \left\{ 1 + \frac{\Delta}{(1-t) - t\Delta} \right\} o^n$$

$$= \frac{t}{(1-t)^2} \left\{ t . o^n + \left(\frac{t}{1-t}\right) \Delta o^n + \left(\frac{t}{1-t}\right)^2 \Delta^2 o^n + \right.$$

$$\left. + \ldots\ldots \left(\frac{t}{1-t}\right)^n \Delta^n o^n \right\}.$$

(21). It is required to prove the following expression for the numbers of Bernoulli,

$$B_{2x-1} = (-1)^{x+1} . \left\{ \frac{\Delta o^{2x+1}}{1^2} - \frac{\Delta^2 o^{2x+1}}{2^2} + \right.$$

$$\left. \ldots\ldots + \frac{\Delta^{2x+1} o^{2x+1}}{(2x+1)^2} \right\},$$

Take the equation (Append. Art. 411.)

$$\Sigma u_x = \int u_x dx - \frac{u_x}{2} + \frac{B_1}{1 . 2} . \frac{d}{dx} u_x - \text{&c.}$$

in which make $u_x = x^{2i}$, and we shall find for the coefficient of x in the developement of $\Sigma (x^{2i})$

$$(-1)^{i+1} B_{2i-1}.$$

But, if we integrate the expression given for x^n in (2. Sect. 2.) we find

$$\Sigma (x^n) = C + (-1)^n . \left\{ \frac{x}{1^2} \Delta o^{n+1} - \frac{x(x+1)}{1 . 2^2} \Delta^2 o^{n+1} + \text{&c.} \right\}$$

If this be developed in powers of x, and the coefficients of x collected together, we shall find for the whole coefficient of that term

$$(-1)^n . \left\{ \frac{\Delta o^{n+1}}{1^2} - \frac{\Delta^2 o^{n+1}}{2^2} + \text{&c.} \right\}$$

In this if we write $2i$ for n, we shall find by comparing it with the coefficient previously found

$$B_{2i-1} = (-1)^{i+1} \left\{ \frac{\Delta o^{2i+1}}{1^2} - \frac{\Delta^2 o^{2i+1}}{2^2} + \&c. \right\}$$

which is, in fact, the equation to be proved. If $2i - 1$ be put for n the term x vanishes form the integral $\Sigma(x^n)$, (See Appendix Art. 369.) which shews that we must have

$$\frac{\Delta o^{2i}}{1^2} - \frac{\Delta^2 o^{2i}}{2^2} + \&c. = 0,$$

which has been already proved by a different process (13. Sect. 7.) It will not be amiss now to notice one more property of the numbers comprised in the form $\Delta^m o^n$ by the aid of which the table of their numerical values given in (33. Sect. 1.) may be continued to any extent, with very little trouble.

(22). The numbers

$$\Delta^n o^n, \ \Delta^n o^{n+1}, \ \Delta^n o^{n+2}, \ \&c.$$

form a recurring series. To shew this, and to determine its scale of relation.

Since (App. Art. 350.)

$$\Delta^n o^r = n^r - \frac{n}{1}(n-1)^r + \ldots \pm \frac{n}{1} . 1^r,$$

this function is of the form

$$C_1 . \alpha^x + C_2 . \beta^x + \ldots C_n . \nu^x.$$

It is, therefore, a particular integral of some equation of differences of the first degree with constant coefficients (App. Art. 389.) and is therefore (App. Art. 390.) the general term of some recurring series. Let now

$$A_0 = 1,$$
$$A_1 = 1 + 2 + 3 + \ldots n,$$

$$A_2 = 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 + \ldots\ldots (n-1) \cdot n$$

. . . .

$$A_n = 1 \cdot 2 \cdot 3 \ldots\ldots n.$$

Then (App. Art. 389.) This equation will be

$$\Delta^n 0^{x+n} - A_1 \Delta^n 0^{x+n-1} + \ldots\ldots \pm A_n \Delta^n 0^x = 0$$

so that the scale of relation is

$$+A_1,\ -A_2,\ +A_3, \ldots\ldots \pm A_n.$$

SECTION IX.

Exercises, &c. in the Interpolation of Series.

(1). In any series of consecutive equidistant values of a function, where one is deficient, to insert that one.

Let the equidistant values be

$$v_0,\ v_1,\ v_2, \ldots\ldots v_n,$$

and let the deficient one be v_i, so that all but v_i are given. Assume $\Delta^n v_0 = 0$, or that the $(n-1)^{\text{th}}$ differences are constant, which will almost always be nearly the case in tabulated results, except under extreme circumstances. Then we have

$$\Delta^n v_0 = v_n - \frac{n}{1} v_{n-1} + \frac{n(n-1)}{1 \cdot 2} v_{n-2} \ldots\ldots \pm \frac{n}{1} v_1 \mp v_0 = 0,$$

an equation of the first degree, from which any one of the values as v_i may be determined in terms of the rest.

(2). Given two values of any function, required to insert one equidistant between them.

Given v_0 and v_2, required v_1,

$$\Delta^2 v_0 = 0, \quad v_2 - 2 v_1 + v_0 = 0,$$

$$v_1 = \frac{v_0 + v_2}{2}.$$

(3). Given three values v_0, v_1, v_3 of any function to insert the deficient one v_2.

$$\Delta^3 u_0 = 0; \quad v_3 - 3 v_2 + 3 v_1 - v_0 = 0,$$

$$v_2 = \frac{v_3 + 3 v_1 - v_0}{3}.$$

In like manner, if v_1 were the deficient value, we should find

$$v_1 = \frac{v_0 + 3 v_2 - v_3}{3}.$$

(4). Given the following common logarithms,

log 510 = 2.70757018
log 511 = 2.70842090
log 513 = 2.71011737
log 514 = 2.71096312

it is required to insert the deficient value log 512.

Given $v_0 = \log 510$, $v_1 = \log 511$, $v_3 = \log 513$, and $v_4 = \log 514$. Required $v_2 = \log 512$.

$$\Delta^4 v_0 = v_4 - 4 v_3 + 6 v_2 + 4 v_1 + v_0 = 0$$

$$v_2 = \frac{4 (v_1 + v_3) - (v_0 + v_4)}{6} = 2.70926996$$

precisely as the table.

(5). In any series of consecutive equidistant values where two are deficient, to insert those two.

As before, let them be

$$v_0, v_1, \ldots \ldots v_{n+1}$$

and assuming $\Delta^n v_0 = 0$ and $\Delta^n v_1 = 0$, to obtain a continuous law of increase or diminution throughout the whole series, we have

$$\left.\begin{array}{l} v_n - \frac{n}{1} v_{n-1} + \frac{n(n-1)}{1.2} v_{n-2} \ldots \pm v_0 = 0 \\ v_{n+1} - \frac{n}{1} v_n + \frac{n(n-1)}{1.2} v_{n-1} \ldots \pm v_1 = 0 \end{array}\right\}$$

two equations of the first degree which suffice to determine any two of the values in terms of the rest. The same principle will serve to insert any number of deficient terms.

(6). Given v_0, v_1, v_4, v_5. Required v_2 and v_3.

Assume $\Delta^4 v_0 = 0$ and $\Delta^4 v_1 = 0$, then

$$\left.\begin{array}{l} v_4 - 4v_3 + 6v_2 - 4v_1 + v_0 = 0 \\ v_5 - 4v_4 + 6v_3 - 4v_2 + v_1 = 0 \end{array}\right\}$$

whence

$$v_2 = \frac{-3v_0 + 10v_1 + 5v_4 - 2v_5}{10}$$

$$v_3 = \frac{-2v_0 + 5v_1 + 10v_4 - 3v_5}{10}.$$

(7). In a table of the values of $\int dx \left(\log \frac{1}{x}\right)^{a-1}$ taken between the limits $x=0$ and $x=1$ *, we find the following values corresponding to the annexed values of a.

$a = 1.326$, $\quad \int = 0.8938710$;
1.328, ... 0.8936220;
1.329, ... 0.8935004;
1.331, ... 0.8932628.

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* Legendre, *Exercises de Calcul Integral.* p. 302.

Required the values corresponding to $a = 1.327$, and [illegible]

Given v_0, v_2, v_3, v_5, required v_1 and v_4.

$$v_1 = \frac{v_5 - 10\,v_3 + 20\,v_2 + 4\,v_0}{15} = 0.8937455$$

$$v_4 = \frac{4\,v_5 + 20\,v_3 - 10\,v_2 + v_0}{15} = 0.8933807.$$

(8). In any series of consecutive equidistant values, where one or more are deficient, and the rest given, to interpolate any intermediate value whatever.

Insert the deficient equidistant values, and then interpolate the series so completed by the formula,

$$v_n = v_0 + \frac{n}{1}\,\Delta\,v_0 + \frac{n\,(n-1)}{1\,.\,2}\,\Delta^2 v_0 + \&c.$$

For instance,

(9). In a table of the values of the function $\tan^{-}(x)$ or arc $(\tan = x)$* we have given

$\tan^{-1} 10 = 1.471127674$, $\tan^{-1} 11 = 1.480136439$

$\tan^{-1} 13 = 1.494024435$, $\tan^{-1} 15 = 1.504228163$.

Required $\tan^{-1}(11.63)$.

The values of $\tan^{-1} 12$ and $\tan^{-1} 14$, first of all inserted are respectively 1.487655094 and 1.499488856.

Let then $n = 1.63$, $v_0 = \tan^{-1} 10$, &c. and we find

$$v_n = 1.4850022707$$

the number required.

(10). Given the values v_0, v_1, v_3 of a function, required to interpolate any given value as v_n,

$$v_n = v_0 - \frac{v_3 - 9\,v_1 + 8\,v_0}{6}\,.\,n + \frac{v_3 - 3\,v_1 + 2v_0}{6}\,.\,n^2.$$

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* Given by Spence in his Logarithmic Transcendents, p. 63.

(11). Two observations of a certain quantity were made at the interval of a day from each other: the first gave for its value a, the second $-b$. When was its value zero?

Given v_0 and v_1, required n so that $v_n = 0$.

1st In general $v_n = v_0 + \frac{n}{1}(v_1 - v_0)$

and making this zero,

$$n = -\frac{v_0}{v_1 - v_0} = \frac{a}{a+b},$$

which is the time in fractions of a day, from the first observation to the moment required. Having but two observations we suppose $\Delta^2 v_0$ &c. zero. This is the most ordinary instance of interpolation. To render it exact, we should, if possible, choose such opportunities for observation, as will allow of our neglecting $\Delta^2 v_0$, &c. that is, when the variation of the quantity observed is nearly uniform. Such is that of the sun's declination near the equinox, of a planet's latitude near its node, &c. The rule for proportional parts in logarithmic and other tables, depends likewise on this problem.

(12). Three observations of a certain quantity were taken at equal intervals. Its value were found to be v_0, v_1, v_2, between which its value was a. When did this happen?

Given v_0, v_1, v_2. Required n so that $v_n = a$, make

$$v_0 + \frac{n}{1}\Delta v_0 + \frac{n(n-1)}{1 \cdot 2}\Delta^2 v_0 = a,$$

whence

$$n = \frac{1}{2\Delta^2 v_0}\Big\{(\Delta^2 v_0 - 2\Delta v_0) + \\ + \sqrt{(\Delta^2 v_0 - 2\Delta v_0)^2 - 8(v_0 - a)\Delta^2 v_0}\Big\}.$$

The positive sign is affixed to the radical, because the supposition $\Delta^2 v_0 = 0$, or $\Delta^2 v_0$ very small ought to reduce the value of n to that given in the last example. If this expression be developed in powers of $\Delta^2 v_0$, and its square and superior powers neglected, we find

$$n = -\left(\frac{1}{\Delta v_0} + \frac{(v_1 - a)\,\Delta^2 v_0}{2\,(\Delta v_0)^3}\right).(v_0 - a)$$

in which we may regard the term

$$-(v_0 - a).(v_1 - a).\frac{\Delta^2 v_0}{2\,(\Delta v_0)^3},$$

as a correction to be applied to the value of n calculated from the terms v_0 and v_1 alone on the supposition $\Delta^2 v_0 = 0$, and thus in certain cases dispense with a troublesome calculation. We may observe that by a proper choice of the quantities represented by v_0, v_1, &c. the quantity a may always be made zero, so that we have

$$-\frac{v_0 . v_1 . \Delta^2 v_0}{2\,(\Delta v_0)^3}$$

for the correction to be made in this case.

(13). Given three values, not equidistant, of a function, to interpolate any intermediate value.

Given v_α, v_β, v_γ, required v_n.

Suppose the *indices* α, β, γ, to be equidistant values of some other function, thus let $\alpha = z_0$, $\beta = z_1$, $\gamma = z_2$ and let $n = z_x$, then will v_α, v_β, v_γ, v_n, be the values of the function v_{z_x} corresponding to the values 0, 1, 2, and x of the index x the independent variable. Let $v_{z_x} = u_x$, then $u_0 = v_\alpha$, $u_1 = v_\beta$, $u_2 = v_\gamma$, $u_x = v_n$, and the formula in Art. 404. gives

$$u_x \text{ or } v_n = \frac{(n-\beta)(n-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}v_\alpha + \frac{(n-\alpha)(n-\gamma)}{(\beta-\alpha)(\beta-\gamma)}v_\beta +$$

$$+\frac{(n-\alpha)(n-\beta)}{(\gamma-\alpha)(\gamma-\beta)}v_\gamma.$$

(14). Three observations of a quantity near its maximum or minimum, are made at given times (equidistant or not.) From the observed values, to determine when the maximum or minimum took place.

When a quantity is near its maximum or minimum, its values cannot be interpolated from two observations, because such interpolation requires the supposition of uniform variation during their interval, which cannot be made in these circumstances. In fact the function $v_0 + \frac{n}{1}(v_1 - v_0)$ does not admit a maximum or a minimum by the variation of n. In the case of three observations, however, suppose α, β, γ, to be the times (from a certain epoch) at which they were made, and v_α, v_β, v_γ, the observed values, then, since at any other time n we have v_n = the expression in the last problem, if we differentiate this relative to n, and put the result $=0$, we shall find

$$n = \frac{(\beta^2 - \gamma^2)v_\alpha - (\alpha^2 - \gamma^2)v_\beta + (\alpha^2 - \beta^2)v_\gamma}{2\{(\beta - \gamma)v_\alpha - (\alpha - \gamma)v_\beta + (\alpha - \beta)v_\gamma\}},$$

the value of n required, at which v_n is a maximum. By this formula may the meridian altitude of the Sun, or a Star, for example, be found when an observation precisely on the meridian cannot be had.

If the observations be equidistant, and the epoch be fixed at the first, we get

$$n = \frac{\beta}{2} \cdot \frac{3v_\alpha - 4v_\beta + v_\gamma}{2v_\alpha - 2v_\beta + v_\gamma}.$$

(15). Given any number of values of a quantity observed at given times, not equidistant, to determine its value and those of its differential coefficients at a given instant, the time being supposed to increase uniformly.

Let the given instant be fixed on for an epoch, and call t the time elapsed since that epoch, t being indeterminate, and negative for all observations preceding the epoch, also let α, β, γ, &c. be the values of t at the moments of observation, and v_α, v_β, v_γ, &c. those of the quantity observed, and we have by Append. Art. 404. for its value v at the time t,

$$v = \frac{(t-\beta)(t-\gamma)\dots}{(\alpha-\beta)(\alpha-\gamma)\dots} v_\alpha + \frac{(t-\alpha)(t-\gamma)\dots}{(\beta-\alpha)(\beta-\gamma)\dots} v_\beta + \&c.$$

The values then of the differential coefficients $\frac{dv}{dt}$, $\frac{d^2v}{dt^2}$, &c, when $t = 0$, or at the epoch, will be the coefficients of t, t^2, &c. in the developement of this function, divided respectively by 1, 1 . 2, 1 . 2 . 3, &c, or calling them $\frac{dV}{dt}$, $\frac{d^2V}{dt^2}$, &c.

$$V = \pm \left\{ \frac{\beta.\gamma.\delta\dots}{(\alpha-\beta)(\alpha-\gamma)\dots} v_\alpha + \frac{\alpha.\gamma.\delta\dots}{(\beta-\alpha)(\beta-\gamma)\dots} v_\beta + \&c. \right\}.$$

$$1 . \frac{dV}{dt} = \mp \left\{ \frac{\beta.\gamma.\delta\dots\left(\frac{1}{\beta} + \frac{1}{\gamma} + \dots\right)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\dots} v_\alpha + \&c. \right\}$$

$$1.2. \frac{d^2V}{dt^2} = \pm \left\{ \frac{\beta.\gamma.\delta\dots\left(\frac{1}{\beta\gamma} + \frac{1}{\beta\delta} + \frac{1}{\gamma\delta} + \dots\right)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\dots} v_\alpha + \&c. \right\}$$

The sign prefixed to the right hand members of these equations is the upper or lower, according as the number of observations, or of the letters α, β, γ, δ, ... is odd or even.

Laplace's method of computing the orbit of a comet, turns upon the application of this problem. The formulæ here deduced are somewhat different in their form from those employed in the Mecanique Celeste, and perhaps, rather more complicated to the eye. But in actual computation they will, I believe, be found more convenient, the terms of which they consist being better adapted to logarithmic computation, and in reality less intricate in their formation, and in consequence affording less room for mistakes on the part of the calculator.

SECTION X.

Application of the Calculus of Differences to the determination of Curves from properties involving consecutive points separated by a finite interval.

(1). In the circle, any line ACB (Part. III. Fig. 1.) drawn through a certain fixed point C (the center of the circle) and meeting the curve at its two extremities, is of a given length in all positions of the line. It is required to determine whether any other curves possess the same property, and if so, to include them under one general equation. In other words: Required the class of curves whose diameters are invariable.

Draw any line CM, and let the angle $MCA=\theta$, $CA=r$, $CB=r'$, and suppose $r=\phi(\theta)$ to be the polar equation of the curve sought. Then will $r'=\phi(\theta+\pi)$ and since by the condition of the question $r+r'=\text{constant}=2a$, we have the following equation for determining the form of ϕ,

$$\phi(\theta)-\phi(\theta+\pi)=2a.$$

Suppose now $z=\frac{\theta}{\pi}$, or $\theta=\pi z$, and this equation becomes

$$\phi(\pi z)+\phi\{\pi.(z+1)\}=2a.$$

This is in fact, an equation of differences; for, if we suppose $\phi(\pi z)=u_z$, we get $\phi\{\pi.(z+1)\}=u_{z+1}$, and

$$u_z+u_{z+1}=2a.$$

Now, this equation deprived of its last term $2a$, is evidently satisfied by $\cos\pi z$, because

$$\cos\pi z+\cos\pi(z+1)=0$$

Hence the complete integral is

$$u_z = a + C \cdot \cos \pi z,$$

C being an arbitrary constant, or rather according to the remark in Appendix, Art. 368. an arbitrary function of $\cos 2\pi z$, or in general, any quantity which does not change by the substitution of $z+1$ for z.

Hence, restoring the original denominations

$$r = a + \cos \theta \,.\, f(\cos 2\theta),$$

where under $f(\cos 2\theta)$ are comprehended all functions of θ, whether algebraic or transcendental, which do not change when $\theta + \pi$ is substituted for θ. Thus if $f(\cos 2\theta) = 0$, $r = a$ the equation of the circle. If $f(\cos 2\theta) = b$, we have

$$r = a + b \cdot \cos \theta,$$

which represents a curve similar to that in the figure, whose algebraic equation is

$$(x^2 - bx + y^2)^2 = a^2 (x^2 + y^2).$$

(2). Instead of supposing the sum of the parts AC, CB, (Part III. Fig. 1.) constant, let their rectangle be invariable. Required the class of curves possessed of this property.

Retaining the same denominations, we have now

$$rr' = a^2, \quad \text{or} \quad \phi(\theta) \,.\, \phi(\pi + \theta) = a^2$$

which treated in the same manner, by supposing $\theta = \pi z$ and $\phi(\theta) = u_z$ gives

$$u_z \,.\, u_{z+1} = a^2$$

which is evidently satisfied by

$$u_z = a \,.\, C^{\cos \pi z},$$

because $\cos \pi (z+1) = -\cos \pi z$. This then, containing an arbitrary constant C is the complete integral, and as before replacing C by an arbitrary function of $\cos 2\pi z$, and restoring the value of z.

$$r = \phi(\theta) = a \,.\, f(\cos 2\theta)^{\cos \theta}.$$

Thus the oval whose equation is $r = a\, e^{\cos\theta}$ satisfies the condition, but it is also satisfied by an infinite variety of algebraic curves, as we shall now shew.

We have already remarked that $f(\cos 2\,\theta)$ may be any function which does not change by writing $\pi + \theta$ for θ. Let $F(\cos 2\,\theta)$ be any other such function, and it is evident that the expression

$$\frac{F(\cos 2\,\theta) + \cos\theta}{F(\cos 2\,\theta) - \cos\theta},$$

by that substitution has its numerator and denominator inverted, because $\cos(\pi + \theta) = -\cos\theta$, hence if this expression be raised to any power such as $\cos\theta$, or $\frac{n}{\cos\theta}$, &c. whose sign only is changed by the substitution, the function so produced will remain unaltered. We are at liberty then to suppose $(f \cos 2\,\theta)$ of the form

$$\left\{\frac{F(\cos 2\,\theta) + \cos\theta}{F(\cos 2\,\theta) - \cos\theta}\right\}^{\frac{n}{\cos\theta}}$$

which value being written for $f \cos 2\,\theta$ in the expression of r above found gives

$$r = a\left\{\frac{F(\cos 2\,\theta) + \cos\theta}{F(\cos 2\,\theta) - \cos\theta}\right\}^{n}$$

which always gives algebraic curves by assigning an algebraic form to the function F. Thus, if we suppose $F(\cos 2\,\theta) = 1$ and $n = -\frac{m}{2}$, we get for the equation of the curve

$$r = a\,.\,(\tan \tfrac{1}{2}\,\theta)^{m}.$$

If we suppose

$$F(\cos 2\,\theta) = \frac{\sqrt{(\alpha^2 - \beta^2\,.\,\sin\theta^2)}}{\beta},$$

which evidently remains unaltered by the substitution of $\pi + \theta$ for θ, we get

$$r = \frac{a}{\sqrt{(a^2-\beta^2)}} \{ \beta . \cos\theta + \sqrt{(a^2 - \beta^2 . \sin\theta^2)} \}$$

which is the equation of a circle, the pole round which the angle θ is reckoned being any point however situated. In fact this property is proved to belong to the circle in the 35th and 36th propositions of the third book of Euclid.

(3). The conchoid is produced by the revolution of a straight line round a fixed pole, one of its points being subjected to move in a straight line, while the other describes the curve. To find a curve, or class of curves, susceptible of being described like the conchoid, with this difference, however, that instead of the *directrix* being a straight line, it shall be another branch of the curve itself to be found.

The straight line CM (Part III. Fig. 2.) is to revolve about C, so that MM' cut off between the intersections M and M', with the two branches AM and BM', shall be constant and $= a$. Draw MP, $M'P'$ perpendicular to CP'. Let $CM=r$, $CM'=r'$, $CP=x$, $PM=y$, $CP'=x'$, $CM'=y$.

Therefore $r' - r = a$, or $r' = a + r$.

Assume $y = \phi\sqrt{(x^2+y^2)} = \phi(r)$ for the equation of the curve. Then, since the same equation is common to both branches, we must have also $y' = \phi(r')$.

Now, by similar triangles CPM, $CP'M'$, we have

$$y' = y . \frac{r'}{r},$$

Therefore

$$\frac{\phi(r')}{r'} = \frac{\phi(r)}{r}, \text{ or } \frac{\phi(a+r)}{a+r} = \frac{\phi(r)}{r},$$

an equation from which the form of ϕ is to be determined. Let $r = az$, then $a + r = a(z+1)$ and if we suppose $\frac{\phi(az)}{az} = u_z$ we have

$$u_{z+1} = u_z,$$

whence, $u_z = f \cos 2\pi z$, and therefore

$$\frac{y}{r} = f\left(\cos 2\pi \, . \, \frac{r}{a}\right).$$

Thus if we suppose $f\left(\cos 2\pi \frac{r}{a}\right) = \sin 2\pi \frac{r}{a}$, and observe that $\frac{y}{r} = \sin MCP = \sin\theta$, we have

$$\sin\theta = \sin 2\pi \frac{r}{a}, \text{ or } r = a \, . \, \frac{\theta}{2\pi},$$

the equation to the spiral of Archimedes, in which the next inferior convolution of the curve, supplies the place of another branch. In fact, the preceding analysis does not take in the condition that M and M' should lie in *different branches*, but the following solution will apply to the strict letter of the enunciation.

Let CM be regarded as a *negative* value of r, answering to one half revolution *more* of the line CM in which case the geometrical equation $r' - r = 2a$ will be represented in analytical language by $\phi(\theta) + \phi(\pi + \theta) = 2a$, and thus the equation of the problem (1) resolves this case, provided we select only such curves as have the property described.

Thus in the result

$$r = a + \cos\theta \, . f(\cos 2\theta),$$

when $\theta < \pi$, r must be positive, and when greater, negative, or at least, if in any part of the variation of θ between o and π, r becomes negative, it must be negative in a higher degree in the corresponding part of its variation between π and 2π. Such a curve is

$$r = a\left(1 + \frac{1}{\sin\theta}\right), \text{ or } (x^2 + y^2)(y - a)^2 = a^2 y^2$$

and an indefinite variety of algebraic curves, among which are some which satisfy the geometrical property $r' + r = 2a$, with one pair of branches, at the same time that $r'-r=2a$ is satisfied by another pair.

(4). Required the nature of the curve AMM' (Part III. Fig. 3.) when the line AMM' revolving round A has the sum of the m^{th} powers of the segments AM, AM' constant, or yet more generally, when one segment $AM' = r'$ is any assigned function $\alpha(r)$ of the other $AM = r$.

As before, suppose $y=\phi(r)$, then we must have $y'=\phi(r')$. Now since $r'=\alpha(r)$ and by similar triangles $\frac{y'}{r'}=\frac{y}{r}$, we have

$$\frac{\phi\alpha(r)}{\alpha(r)}=\frac{\phi(r)}{r},$$

an equation for determining the form of the function ϕ. Now it is evident that if any function $f(r)$ can be found which does not change when $\alpha(r)$ is written for r, the equation

$$\frac{\phi(r)}{r}=f(r), \text{ or } \phi(r)=r\,.f(r)$$

satisfies the above, now Laplace's method explained in Appendix Art. 398.) affords a general solution of the equation

$$f\{\alpha(r)\}=f(r),$$

and thus the complete solution of the problem may be had. In the particular case proposed, however, the function $\alpha(r)$ is one of a very singular class of functions, which render the application of Laplace's method extremely delicate, and moreover unnecessary. It will be observed that since $r^m+r'^m=a^m$, therefore

$$r'=\alpha(r)=\sqrt[m]{(a^m-r^m)}.$$

Hence we have $\alpha(r')=\alpha\alpha(r)=\alpha^2(r)=\sqrt[m]{a^m-(a^m-r^m)}=$ $=r$. The function in question is therefore one of those which may properly be called *periodic functions*, under which may be comprehended all which satisfy such equations as

$a^2(r) = r$, $a^3(r) = r$, &c. and which are possessed of a variety of the most elegant and useful properties, which this is not the place to enumerate. However, it is here to our purpose to remark, that any symmetrical function of r and $a(r)$ has the property we wish, viz. that it does not change by the substitution of $a(r)$ for r, because r becoming $a(r)$, and $a(r)$, $a^2(r)$ or r, these quantities only change places by this substitution, which, as they are similarly involved, does not alter the value of the function. Let us for instance suppose (in the proposed case)

$$f(r) = \frac{r}{b^2}a(r) = r . \frac{\sqrt[m]{(a^m - r^m)}}{b^2},$$

and we find

$$b^2 y = r^2 . \sqrt[m]{(a^m - r^m)}.$$

Similarly, the equations

$$b . \left(\frac{y}{r}\right)^n = r + \sqrt[m]{(a^m - r^m)};$$

and
$$b y = a^m r^{m+1} - r^{2m+1}, \text{ \&c.}$$

may be shewn to satisfy the condition of the problem.

The cases where $r + r' = a$, $r^m + r'^m = a^m$, $r^2 . r' = a^3$, were proposed long ago in the Leipsic Acts by John Bernoulli, at the same time with the celebrated problem of the Brachystochrone, as a defiance to the mathematical world; but it does not appear that their real object, or the point where their difficulty rested was perceived, either by their proposer, or by any one of the numerous and eminent geometers who gave solutions of them. The attention of mathematicians being, however, immediately occupied by the extraordinary controversy of the Bernoullis, and the discoveries of James relative to Isoperimetrical problems, to which that of the Brachystochrone gave rise, the present questions were allowed to sink into a degree of oblivion, from which, it will not be amiss if we attempt to rescue them. They were proposed, as J. Bernoulli expressly states, with a view of calling the attention of geometers, to a case where the Cartesian methods of reducing the conditions of a geometrical problem to an equation entirely failed, while at the same time the differential calculus afforded no assistance; thus presenting a difficulty which seemed quite unexpected, and of a different kind from any which had yet been felt. This difficulty is in fact the solution of a *functional equation*, or the determination of an unknown function from an equation, such as those of (Prob 4.) where it enters under more forms than one, but Leibnitz, L'Hopital, Newton, and Jas. Bernoulli, all of whom resolved the problems *, were contented with the first particular forms of the unknown function which presented themselves, without attempting to discover any direct process by which the functional equation might be resolved, and which in cases of a little greater complexity, constitutes the only analytical difficulty to be surmounted. It is rather surprising that this was not observed by Jas. Bernoulli, who distinctly reduces the problem where $r \cdot r' = a^2$ to the determination of the form of an equation, which shall remain unaltered by certain changes made among the variables it contains. His solution of the problem which requires that $r^2 \cdot r'$ shall be invariable, is erroneous, and for a very obvious reason, the neglect of the constant in the integral $\int \frac{dx}{x \cdot \log x}$, and indeed he himself calls his solution "dubiæ et suspectæ veritatis."

The subject was resumed by Clairaut in 1734, in a memoir communicated to the Academy of Paris, in which he resolves several problems by a method professedly grounded on, and equivalent to that employed by Newton in the solu-

* Newton's solution, though extremely elegant, turns on a peculiarity in the case proposed. It is an application of one of his own discoveries respecting the sums of the powers of roots of an equation, and a very happy one, but the question seems not to have struck him in the light we are now considering it.

tion of Bernoulli's problem above-mentioned, developed, however, with great ingenuity, and applied in particular to one problem of no ordinary difficulty, "To determine the nature of a curve, such that the intersection of any two of its tangents which include a given angle, shall always be found in a given curve." It was in the solution of this problem, that Clairaut first discovered the class of differential equations treated of in (Art. 270. of the text) whose general form is

$$y - x\frac{dy}{dx} = f\left(\frac{dy}{dx}\right),$$

and which has procured him with some, the unmerited praise of having first discovered the particular solutions of differential equations; our countryman Brook Taylor in 1715 having deduced the same conclusion in the same manner, and made the same observation on it *, in integrating the equation

$$\left(y - x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2,$$

which is evidently a particular case of Clairaut's general form. Since that period many geometers have occupied themselves with the solution of problems of the kind in question, remarkable examples of which may be found in the writings of Euler, Voss, and Biot.

(5). To determine a class of curves possessed of the following property: viz. that supposing a system of lines, n in number, originating in a fixed point, and terminating in the curve, to revolve about this point, making always equal angles with each other, their sum shall be invariable.

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* He differentiated, and obtained an equation composed of two factors, one of which leads to a final result free from differentials, but containing no arbitrary constant, which is, says he "Singularis quædam solutio problematis." See a clear and impartial statement of the whole in Lagrange's Leçons sur le Calcul des fonctions. Lect. xvii.

The angle made by one of them (r) with some fixed line being θ, those made by the others will be respectively

$$\theta + \frac{2\pi}{n},\ \theta + \frac{4\pi}{n}, \dots \theta + \frac{2(n-1)\pi}{n}.$$

Consequently, if $r = \phi(\theta)$ we must have

$$\phi(\theta) + \phi\left(\theta + \frac{2\pi}{n}\right) + \dots \phi\left(\theta + \frac{2(n-1)\pi}{n}\right) = n\,a,$$

$n\,a$ being some given quantity. Suppose $\theta = \frac{2\pi z}{n}$, then $\theta + \frac{2\pi}{n} = \frac{2\pi.(z+1)}{n}$, and so on. If then we make $\phi\left(\frac{2\pi z}{n}\right) = u_z$, we have

$$u_z + u_{z+1} + \dots u_{z+n-1} = n\,a.$$

The several particular integrals of this equation deprived of its constant term are

$$\cos\frac{2\pi z}{n},\ \cos\frac{4\pi z}{n}, \dots \cos\frac{2(n-1)\pi z}{n}$$

for the sum of the series

$$\cos A + \cos(A+B) + \dots \cos\{A + (n-1)B\}$$

being

$$\cos\left\{A + \frac{n-1}{2}B\right\}.\frac{\sin\left(\frac{n}{2}B\right)}{\sin\left(\frac{1}{2}B\right)}$$

vanishes whenever $\frac{nB}{2}$ is a multiple of π. Now, if either of the above cosines be put for u_z in the expression

$$u_z + u_{z+1} + \dots u_{z+n-1},$$

a series of this form will arise. These functions then severally satisfy the equation

$$u_z + u_{z+1} + \dots u_{z+n-1} = 0.$$

Of course the complete value of u_z in the proposed, or of $\phi(\theta)$ is

$$u_z = a + C_1 . \cos \frac{2\pi z}{n} + C_2 . \cos \frac{4\pi z}{n} +$$

$$\dots . C_{n-1} . \cos \frac{(2n-2)\pi z}{n};$$

but $\theta = \frac{2\pi z}{n}$, also C_1, C_2, &c. may be arbitrary functions of $\cos 2\pi z$, that is, of $\cos n\theta$.

Let them be represented by

$$f_1(\cos n\theta),\quad f_2(\cos n\theta), \text{ \&c.}$$

then

$$\phi(\theta) = r = a + \cos\theta . f_1(\cos n\theta) + \cos 2\theta . f_2(\cos n\theta) +$$

$$\dots\dots + \cos(n-1)\theta . f_{n-1}(\cos n\theta).$$

It is easy then to assign an unlimited variety of algebraic curves which answer the condition. The simplest is that whose equation is

$$r = a + b . \cos\theta,$$

which we have already noticed in the case of $n = 2$, and it is a very remarkable property of this curve that it answers for every value of n. In other words: "In the curve whose "equation is

$$(x^2 - bx + y^2)^2 = a^2(x^2 + y^2)$$

"if a system of any number of radii terminating in the curve, "and making equal angles with each other, be made to revolve "round the origin of the co-ordinates, their sum will be "invariable throughout the whole extent of the curve."

(6). In the parabola, any straight line being drawn through the focus meeting the curve both ways, the tangents at its two extremities include a right angle. To what class of curves does this property, viz. that two tangents so drawn shall include a *given* angle, belong.

Let P and P' (Part III. Fig. 4.) be the two points in the curve, PTQ, $P'QT'$, the tangents; MP, $M'P'$, ordinates. $SM=x$, $SM'=x'$, $SP=r$, $MSP=\theta$, $MP=y$, $M'P'=y'$.

Then, since $\angle PQP'=QTT'+QT'T=PTS+P'T'S$, and that $\tan PTS=\frac{dy}{dx}$, and $\tan P'T'S=-\frac{dy'}{dx'}$, because it lies on the contrary side of the axis of the abscissæ, therefore if we suppose $PQP'=A$, we have

$$\tan A=\frac{\frac{dy}{dx}-\frac{dy'}{dx'}}{1+\frac{dy}{dx}\cdot\frac{dy'}{dx'}} \qquad (a).$$

Now, the points P, P', lying both in the curve, $\frac{dy'}{dx'}$ must be the same function of $\theta+\pi$ that $\frac{dy}{dx}$ is of θ, because the same equations belong to both, hence if we suppose

$$\frac{dy}{dx}=\phi(\theta), \text{ we have } \frac{dy'}{dx'}=\phi(\pi+\theta)$$

and supposing $z=\frac{\theta}{\pi}$ and $u_z=\phi(\theta)$ the equation (a) becomes

$$\tan A=\frac{u_z-u_{z+1}}{1+u_z u_{z+1}},$$

or

$$u_{z+1}u_z-\text{cotan } A\,.(u_{z+1}-u_z)+1=0.$$

This equation we have already integrated (Sect. 4. 20.) and by applying the formula there obtained, we get

$$u_z=\tan(Az+\tan^{-1}C)$$

that is, C being replaçed as before by an arbitrary function of $\cos 2\pi z$,

$$\phi(\theta)=\tan\left\{\frac{A}{\pi}\theta+\tan^{-1}f(\cos 2\theta).\right\}; \qquad (b).$$

Now we have $y = r \cdot \sin\theta$, and $x = r \cdot \cos\theta$, which gives

$$\frac{dy}{dx} = \frac{dr \cdot \cos\theta + r\,d\theta \cdot \cos\theta}{dr \cdot \cos\theta - r\,d\theta \cdot \sin\theta}$$

which put equal to $\phi(\theta)$ gives a differential equation between r and θ, viz.

$$\frac{dr}{r} = \frac{\cos\theta + \phi(\theta) \cdot \sin\theta}{\phi(\theta) \cdot \cos\theta - \sin\theta}\,d\theta.$$

Now the equation (b) gives

$$\phi(\theta) = \frac{\tan\frac{A}{\pi}\theta + f(\cos 2\theta)}{1 - \tan\frac{A}{\pi}\theta \cdot f(\cos 2\theta)}.$$

This value of $\phi(\theta)$ substituted in the differential equation gives after all reductions

$$\log\frac{r}{a} = \int d\theta \cdot \cot\left\{\tan^{-1} f(\cos 2\theta) - \frac{\pi - A}{\pi}\theta\right\}$$

which is the polar equation of the curve sought. Suppose, to take a particular case, $f(\cos 2\theta) = 0$, and we have

$$\log\frac{r}{a} = \int d\theta \cdot \cot\left(\frac{A - \pi}{\pi}\theta\right)$$

$$= \frac{\pi}{A - \pi} \cdot \log\sin\left(\frac{A - \pi}{\pi}\theta\right)$$

and consequently

$$r = a \cdot \left\{\sin\frac{A - \pi}{\pi}\theta\right\}^{\frac{\pi}{A - \pi}}$$

which always gives algebraic curves when the angle A included between the tangents is commensurate to the whole circumference. Thus if $A = \frac{\pi}{2}$, we have $\frac{A - \pi}{\pi} = -\frac{1}{2}$, and

$$r = \frac{a}{\left(\sin\frac{\theta}{2}\right)^2},$$

the equation of a parabola, the origin of the co-ordinates being in the focus.

If $A = \pi$, or the tangents at opposite extremities of the line are parallel, the general equation gives

$$\log \frac{r}{a} = \int d\theta \,.\, f(\cos 2\theta),$$

because the form of the function f being arbitrary we may replace cot $\tan^{-1} f(\cos 2\theta)$ by $f(\cos 2\theta)$ without infringing on the generality of the equation. The equation in this form includes, 1st the logarithmic spiral, by making $f(\cos 2\theta) = b$, and 2dly all curves consisting of four similar parts arranged in the manner of quadrants round a center. The reader will find other solutions of this problem by Messrs. Wallace and Ivory in Leybourne's Repository, (New Series Quest. 172.) which are well worthy his attention.

Euler, and more lately Ivory, in another solution of this problem (Thomson's Annals of Philosophy, Oct. 1816.) have shewn that it admits no solution unless in the cases when the tangents are parallel or include a right angle, but this limitation arises from the assumed condition that the straight line PSP' shall not cut the curve in more than two points. If we admit, however, that the points P, P', may lie in different branches of the same curve, our solution above will apply. Mr. Ivory's final equation is (in the general case) a functional equation of the form $F(\phi x, \phi \alpha x) = 0$, in which $\alpha^2 x = x$, and where the function F is not symmetrical. This he properly remarks is an impossible equation. If, however, we admit that different values of the function ϕ arising from radicals, &c. involved in it, may be used in ϕx and in $\phi \alpha x$, the impossibility vanishes and the equation may be satisfied. This remark on the nature of such equations abstractly considered, is due to Mr. Babbage. The problem just solved affords an illustration of its geometrical signification.

(7). In the parabola, the two tangents PQ, $P'Q$, (See Part III. Fig. 4.) always intersect in the directrix. To generalize this property, or to find a class of curves such that tangents drawn at opposite extremities of any line PSP' passing through a given point S shall always meet in a straight line given in position.

Let AQ be the straight line, and let SA perpendicular to it be taken as the axis of the x, then retaining the construction and denominations of the last problem we have, supposing $SA = X$ and $AQ = Y$.

$$ST = x - y \cdot \frac{dx}{dy}; \quad AT = X - x + y \cdot \frac{dx}{dy},$$

Consequently $AQ = -\frac{dy}{dx} \cdot AT$, or

$$Y = -\frac{dy}{dx}(X - x) - y; \qquad (a).$$

Similarly we should obtain

$$Y = -\frac{dy'}{dx'}(X - x') - y',$$

and equating these we find

$$X = \frac{\left(x'\frac{dy'}{dx'} - x\frac{dy}{dx}\right) - (y' - y)}{\frac{dy'}{dx'} - \frac{dy}{dx}},$$

Now the condition of the problem requires that this shall be constant. Denoting it then by $2a$, we get

$$2a\left(\frac{dy'}{dx'} - \frac{dy}{dx}\right) = x'\frac{dy'}{dx'} - x\frac{dy}{dx} - (y' - y),$$

or,

$$(2a - x') \cdot \frac{dy'}{dx'} + y' = (2a - x) \cdot \frac{dy}{dx} + y.$$

Suppose now

$$(2a-x)\cdot\frac{dy}{dx}+y=u_z,$$

z being some quantity, which changes to $z+1$ when x, y, change to x', y', then we shall have

$$u_{z+1}=u_z, \text{ or } \Delta u_z=0$$

and of course

$$u_z=\text{constant}.$$

But since the points P, P' are so related that (by similar triangles)

$$\frac{y'}{x'}=\frac{y}{x},$$

Therefore the function $\frac{y}{x}$ does not change by the change of x, y, to x', y', or of z to $z+1$, hence the constant in the above equation may be a function of $\frac{y}{x}$ and denoting this by $f\left(\frac{y}{x}\right)$, we have

$$(2a-x)\cdot\frac{dy}{dx}+y=f\left(\frac{y}{x}\right); \qquad (b).$$

This equation is integrable at once, by putting $\frac{y}{x}=u$ which gives

$$\frac{dx}{x(2a-x)}+\frac{du}{2au-f(u)}=0,$$

whence

$$\log\frac{x}{2a-x}+2a\cdot\int\frac{du}{2au-f(u)}=\log c, \qquad (c).$$

Suppose, for instance, $f(u) = -\frac{2a}{u}$; then we have

$$\log \frac{x}{2a - x} + \frac{1}{2} \log (u^2 + 1) = \log c,$$

$$\frac{x^2}{(2a - x)^2}(u^2 + 1) = c^2,$$

or, replacing the value of u and reducing

$$x^2 + y^2 = c^2 (2a - x)^2,$$

which is the general equation of a conic section about the focus, and it is easily seen that the straight line in which the tangents meet, is no other than what is called the directrix in some treatises on conic sections. The conic section itself will be an ellipse, parabola or hyperbola, according as the angle between the tangents is acute, right, or obtuse.

The conic sections also satisfy the conditions of the problem in another way, which, taken in conjunction with what has just been proved, may be considered as affording a very elegant property of these curves.

If we assume $f(u) = + \frac{2a}{u}$, we find

$$\log \frac{x}{2a - x} + \frac{1}{2} \log (u^2 - 1) = \log c,$$

which reduced, as before, gives

$$y^2 - x^2 = c^2 (2a - x)^2.$$

This is likewise the general equation of the conic sections, but whereas in the former case the origin of the co-ordinates was in the focus, and the straight line in which the tangents meet, the directrix; in this it is just the reverse. The origin of the co-ordinates being now in the intersection of the axis with the directrix, and the tangents meeting always in a line drawn through the focus at right angles to the axis (that is, in the *latus rectum* indefinitely produced.) The reader may consult the Mathematical Repository, iii. p. 39. Quest. 267. for another solution of this question by Mr. Lowry.

The final equation (*c*) of this problem presents a peculiarity which ought to be remarked. It is evident that by properly assuming the form of the function $f(u)$, the integral in the first member may be made to have any form we please, and therefore the equation may express any conceivable relation between x and u, or x and y. Yet it is equally obvious, that it is not every possible curve which satisfies the conditions of the problem, but only those of a certain class. The function $f(u)$ then cannot be absolutely arbitrary, but must be subject to certain limitations. Nevertheless, if we recur to the equation (*b*) in which the function f was first introduced, we see no obvious reason for admitting any limitation of its generality; for the first member is merely the analytical expression for the distance AQ (which is easily proved) and as the second remains unaltered so long as the ratio of x to y remains the same, or the point P' lies in the same straight line with P and S, this equation appears to be nothing more than a mere translation of the condition of the question into algebraic language. The elucidation of this delicate point depends upon the theory of eliminations.

Whatever may be the nature of a curve, if we put $\frac{y}{x} = u$, we may eliminate either y or x between this equation and that of the curve, and thus both x and y are expressible in functions of u. For the same reason $\frac{dy}{dx}$ is so expressible, and therefore the first member of (*b*) is *in all cases* reducible, by the theory of elimination to a function of u or $\frac{y}{x}$. But it does not thence follow that every curve possesses the property in question, for this function may have several values, and in all the excepted cases actually has so. It appears therefore that we are not at liberty, in assuming $f(u)$, to substitute any form susceptible of more than one value, but provided this limitation be attended to, it is in all other respects arbitrary. If we put therefore for f, only rational functions, we are always sure to arrive at satisfactory solutions, but in all other cases it is indispensably necessary to *try* the solution obtained before it can be relied on. Considerations of this, or a similar kind, apply to most problems of the nature now under examination, and will obviate any objections arising from the necessity of limiting functions which seem at their first introduction perfectly arbitrary.

(8). Required the class of curves which possess the following property, that any ordinate PM (Part III. fig. 5.) being erected, and normal MP_1 drawn, and at the foot of this normal, another ordinate P_1M_1 erected, and another normal M_1P_2 drawn, and so on, then the subnormals PP_1, P_1P_2, P_2P_3, &c. shall all be equal to each other in the same series, however they may differ in different series, arising from a different position of the first ordinate.

Let the abscissæ AP &c. &c. be represented by x_0, x_1, x_2, &c. the general term x_z being some certain unknown function of the rank it holds in the series or of z, and let the ordinates be y_0, y_1, y_2, &c. Then the subnormal

$$PP_1 = y_0 \cdot \frac{dy_0}{dx_0}; \quad P_1P_2 = y_1 \cdot \frac{dy_1}{dx_1}; \text{ and so on}$$

hence,

$$x_1 = x_0 + y_0 \cdot \frac{dy_0}{dx_0}; \quad x_2 = x_1 + y_1 \cdot \frac{dy_1}{dx_1},$$

and, in general, whatever be z.

$$x_{z+1} = x_z + y_z \cdot \frac{dy_z}{dx_z},$$

or

$$\Delta x_z = y_z \cdot \frac{dy_z}{dx_z}.$$

Now, by the condition of the problem, the series of subnormals

$$y_0 \frac{dy_0}{dx_0}, \quad y_1 \frac{dy_1}{dx_1}, \text{ \&c.}$$

are all equal, therefore this equation is to be integrated on the hypothesis of $y_z \cdot \frac{d y_z}{d x_z}$ being invariable by the change of z to $z+1$. It may therefore be supposed an arbitrary function of $\cos 2\pi z$ or (which comes to the same thing) of $\tan \pi z$, which let us call Z, then,

$$\Delta x_z = Z, \text{ and } \Delta Z = 0,$$

whence integrating

$$x_z = Zz + Z'$$

Z' being another quantity of the same kind, or another arbitrary function of $\tan \pi z$. But we have

$$y_z \cdot \frac{d y_z}{d x_z} = Z,$$

and from these equations, we have only to eliminate z and we get a differential equation between x_z and y_z (or as we will now call them, x and y) expressing the nature of the curve. Now this is easy: for since both Z and Z' are arbitrary functions of $\tan \pi z$, we may suppose one an arbitrary function of the other.

Let then

$$Z' = f(Z),$$

and our equations become

$$x = Zz + f(Z),$$

$$y \frac{dy}{dx} = Z.$$

The first gives

$$z = \frac{x - f(Z)}{Z},$$

or for Z substituting its value given by the second

$$z = \frac{dx}{y\,dy}\left\{x - f\left(\frac{y\,dy}{dx}\right)\right\}.$$

Now Z is an arbitrary function of $\tan \pi z$; let then

$$Z = F(\tan \pi z)$$

In this equation for Z write its value $y\frac{dy}{dx}$, and for z its value just determined, and we get

$$y\frac{dy}{dx} = F\left\{\tan \pi\left(\frac{x - f\left(y\frac{dy}{dx}\right)}{y\frac{dy}{dx}}\right)\right\}$$

the differential equation of the curve required, which we see involves two arbitrary functions. In the particular case where F denotes an absolute constant c, we have

$$y\,dy = c\,dx, \quad y^2 = 2\,c\,x + c'$$

the equation of a parabola.

(9). At any point of a certain curve, let a normal be drawn and an ordinate erected, let a second ordinate be taken equal to the first subnormal, and let a second normal be drawn. Required the nature of the curve, that the second subnormal so determined shall be equal to the first ordinate, in other words, that in any part of the curve constructing a triangle whose hypothenuse is the normal, and sides the ordinate and subnormal, if this be turned into a subcontrary position and adjusted to fit the curve, its hypothenuse shall still be a normal.

The subnormal being $y\frac{dy}{dx}$, we have the second ordinate y' equal to $y\frac{dy}{dx}$. Suppose now we take

$$y\frac{dy}{dx} = \phi(y).$$

Then will the second subnormal, or

$$y'\frac{dy'}{dx'} = \phi(y') = \phi\left(\frac{y\,dy}{dx}\right)$$

$$= \phi\,\{\phi(y)\} = \phi^2(y).$$

[illegible] by the condition of the question this is equal to the [illegible] ordinate y, hence

$$\phi^2(y) = y,$$

a functional equation from which the nature of the function ϕ is to be determined.

Take

$$y = u_z, \text{ and } \phi(y) = u_{z+1}$$

then we have

$$u_{z+1} = \phi u_z$$

but the proposed equation gives

$$\phi^2(y) = (\phi u_{z+1} =) y = u_z.$$

Hence we have by subtraction

$$\phi u_{z+1} - \phi u_z = -(u_{z+1} - u_z)$$

or,

$$\Delta \phi u_z = - \Delta u_z$$

whence,

$$\phi u_z + u_z = \text{constant}.$$

But the two equations

$$u_z = \phi u_{z+1}, \text{ and } \phi u_z = u_{z+1}$$

give by cross multiplication

$$u_z \cdot \phi u_z = u_{z+1} \cdot \phi u_{z+1}.$$

This function, therefore, does not change by the change of z to $z+1$, and the constant may therefore be an arbitrary function of it, so that

$$u_z + \phi u_z = f(u_z \cdot \phi u_z),$$

or

$$y + \phi(y) = f\{y \cdot \phi(y)\}$$

from which $\phi(y)$ may be found by the ordinary analysis, any form we please being assigned to f.

Now, integrating the equation,

$$\frac{y\,d\,y}{d\,x} = \phi\,(y)$$

we find

$$x = \int \frac{y\,d\,y}{\phi\,(y)},$$

the equation of the curve. Thus if we suppose $f\,(y\,.\,\phi\,y) = a + b\,y\,.\,\phi\,(y)$ we have

$$y + \phi\,(y) = a + b\,y\,.\,\phi\,(y)$$

$$\phi\,(y) = \frac{y - a}{b\,y - 1}$$

$$x = \int \frac{b\,y^2 - y}{y - a}\,d\,y.$$

If $a = b = 1$, we have $x = \int y\,d\,y = \frac{y^2}{2}$, the equation of a common parabola, which therefore satisfies the problem, as does also the cubic parabola.

(10). Required the nature of the curve, such that an ordinate being drawn to any point, and also a radius of curvature, a second point may always be found so related to the first, that the ordinate at the second point shall be equal to the radius of curvature at the first, and the radius of curvature at the second equal to the ordinate at the first.

This problem, by supposing

$$\frac{(d\,x^2 + d\,y^2)^{\frac{3}{2}}}{-\,d\,x\,d^2\,y} = \phi\,(y)$$

leads to precisely the same equation

$$\phi^2\,(y) = y$$

from which determining $\phi\,(y)$, the differential equation above given suffices to determine the curve.

If we suppose $\phi(y) = \frac{a^2}{y}$ which evidently satisfies the condition, we have

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \frac{d^2 y}{dx^2} \cdot \frac{a^2}{y},$$

or, putting $\frac{dy}{dx} = p$,

$$y\,dy = \frac{a^2 p\,dp}{(1+p^2)^{\frac{3}{2}}}$$

$$y^2 - b = -\frac{2a^2}{\sqrt{(1+p^2)}},$$

whence, restoring the value of p

$$dx = \frac{(y^2 - b)}{\sqrt{\{4a^4 - (y^2 - b)^2\}}} \cdot dy$$

the equation of an elastic curve. And in the very same manner might the problem be resolved, if instead of the ordinate and radius of curvature, we had taken any other pair of lines expressible in terms of y, $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, &c. the equations to be resolved being, first, the functional equation of the second order $\phi^2(y) = y$, and secondly, a differential equation in which the function so determined is involved. It would be easy to multiply examples of this kind, but what we have already given will suffice to indicate the method to be pursued in more difficult enquiries of the same nature. They all lead to functional equations of greater or less complexity the solution of which is sometimes easily accomplished by reducing them to equations of differences, though more frequently by considerations peculiar to themselves. This problem and similiar ones may be resolved also by a consideration of the following kind. It is evident that since the radius of curvature at the second point is equal to the ordinate at the first, and the radius of curvature at the first to the ordinate at the second, these two functions (the radius of curvature and the ordinate regarded as functions of x the abscissa) must be such, that when the ordinate changes to the radius of curvature, the radius of curvature shall change to the ordinate, and therefore any symmetrical function of them will remain unchanged. Let f be the characteristic of such a function, then if R be the radius of curvature, it is evident that

$$f(y, R) = \text{constant}$$

will satisfy the condition, because this change being made the equation becomes

$$f(R, y) = \text{constant},$$

which by the peculiar form of f is identical with the former. For R now write its value and we get

$$f\left(y, \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-dx\,d^2y}\right) = a$$

for the equation of the curve. Thus if we assume

$$y \,.\, R = \text{constant}, \quad R = \frac{\text{const.}}{y}$$

which is the case just resolved.

(11). To determine a curve, such that a moveable point setting off from a given place shall return to the same place after being twice reflected against the curve (the angle of incidence being supposed equal to that of reflection) in whatever direction it first sets off.

Let PAP' (Part III. Fig. 6.) be the curve required, S the given point then if $SPP'S$ be the course of the moving point and tangents, &c. be drawn, we have, supposing $SM = x$, $MP = y$,

$$\tan SPM = \frac{x}{y}$$

$$\tan MPT = \frac{dx}{dy}, \text{ } y \text{ decreasing as } x \text{ increases,}$$

whence we get

$$\tan SPT = \tan (MPT - SPM)$$
$$= \frac{\frac{y}{x} - \frac{dy}{dx}}{1 + \frac{y}{x} \cdot \frac{dy}{dx}} = \tan VPH,$$

or, putting $\frac{dy}{dx} = p$,

$$\tan VPH = \frac{y - px}{x + py}$$

by the condition of the first reflexion. But we also have,

$$\tan VPM = -\frac{dx}{dy},$$

and therefore

$$\tan HPM = \tan (VPH - VPM)$$
$$= \frac{x(1 - p^2) + 2py}{y(1 - p^2) - 2px}.$$

But, if we draw $P'N$ perpendicular to PM produced, we have $\tan HPM = \tan P'PN = \frac{MM'}{PM + P'M'} = \frac{x - x'}{y - y'}$, y' being made negative because it lies on the other side of the axis. Hence we must have

$$y' - y = (x' - x) \cdot \frac{y(1 - p^2) - 2px}{x(1 - p^2) + 2py}; \qquad (a).$$

If we regard y, y', and x, x', as the successive values of two functions y and x, of a certain independent variable z, which changes to $z + 1$ when the point P changes to P', and if we suppose the fraction in the second member of this equation equal to P the equation becomes

$$\Delta y = P \Delta x; \qquad (b).$$

This equation alone is not sufficient, as it contains two unknown functions x and y. To obtain another we must consider that by what was before proved

$$P = \text{cotan } HPM,$$

and therefore P' (the value of P corresponding to $z+1$) will be cotan $HP'M'$. But the condition of the question requires P, H, and P' to lie in one straight line, consequently $HPM = HP'M'$ and

$$P' = P, \ P' - P = \Delta P = 0,$$

which integrated gives

$$P = \text{const} = \text{funct} (\tan \pi z) = Z; \qquad (c).$$

Substituting this in (b) it is integrable at once and gives

$$y = Z . x + Z',$$

Z' being another invariable function of z. Now our object being only to obtain a final equation between x and y, we have only to eliminate the auxiliary variable z, with which we have no farther concern, between the equations

$$y = x . Z + Z'$$

$$Z = \frac{y(1-p^2) - 2px}{x(1-p^2) + 2py}$$

in which we shall succeed by the same artifice as in Prob. 8. of this section: for Z and Z' being both arbitrary functions of $\tan \pi z$, we may suppose one an arbitrary function of the other, or $Z' = f(Z)$ when we have

$$y = x . Z + f(Z)$$

in which substituting for Z its value

$$y = x . \frac{y(1-p^2) - 2px}{x(1-p^2) + 2py} + f\left(\frac{y(1-p^2) - 2px}{x(1-p^2) + 2py}\right)$$

which is the differential equation of the curve.

If the arbitrary function be assumed equal to zero, the equation is that of a circle; and if constant, that of a conic section about the focus. In the former case no integration is required.

If we consider attentively the above solution, we shall see that it contains the general principle on which that of all such problems depends. Let us therefore take up the question generally.

(12). To determine the nature of a curve from any property whatever connecting two of its points separated by a finite interval. (See the figure of the last problem.)

Whatever be the nature of the property, it must enable us when one of the points P is assumed, and the figure of the curve known, to determine the other, P', or the direction of the line PP', hence the condition of the problem always enables us to express the cotangent of the angle MPH by some function more or less complicated of the co-ordinates at the points P, P', and their differential coefficients such as

$$F\left(x, y, x', y', \frac{dy}{dx}, \frac{dy'}{dx'}, \text{\&c.}\right)$$

which we will call P.

Hence by a reasoning precisely similar to that of the last problem we get

$$y' - y = P\,.\,(x' - x)\,;\ \text{or}\ \Delta y = P\,.\,\Delta x.$$

But since the property is common to the two points and connects them with each other, the same co-tangent determined by setting out from the point P' will be represented by P' and may be had from P by changing x, y, to x', y', and *vice versâ*. But cotan $M'P'H =$ cotan MPH, so that

$$P' = P,\ \text{or}\ \Delta P = 0.$$

Consequently P must be regarded as invariable, and we have as before

$$y = P x + f(P) \ldots\ldots (a).$$

The problem now divides itself into two cases, 1st when the function P involves only the co-ordinates of the point P and their differentials; and 2dly, when those of both the points concerned are combined in it. In the former case the equation (a) containing only x, y, and their differentials, is itself the final differential equation of the curve sought. In the latter, however, another process is requisite. The equations

$$y' - y = P(x' - x),$$

$$y = P x + f(P),$$

$$P' = P,$$

must be combined to eliminate both x' and y' and the resulting equation will express the nature of the curve.

(13). For instance, suppose the relation of the two points P and P' such that a line drawn perpendicular to the curve at either of them, shall pass through the other.

Here $P H$ is a normal to the curve, and therefore,

$$P = \tan P H M = \frac{d x}{-d y} = \frac{-1}{p}, \quad \left(\text{putting } p = \frac{d y}{d x}\right)$$

Therefore the differential equation to the curve is

$$y = \frac{-x}{p} + \text{funct}\left(\frac{-1}{p}\right),$$

or

$$x + p y = f(p).$$

If we take $f(p) = 0$, we have

$$x\,d\,x + y\,d\,y = 0,$$

$$x^2 + y^2 = c^2$$

the equation of a circle.

(14). Required the nature of the curve in which if $M H'$ be always taken equal to $\frac{x\,y + a\,(y' + b)}{a^2} \times y$; the points P, P' shall be convertible, that is, that $M' H$ shall equal $\frac{x'\,y' + a\,(y + b)}{a^2} \times y'$.

Since $P = \frac{M\,H}{M\,P}$, we have $P = \frac{x\,y + a\,(y' + b)}{a^2}$;

therefore the equation $P' = P$ gives

$$x\,y + a\,y' + a\,b = x'\,y' + a\,y + a\,b,$$

$$a\,(y' - y) = x'\,y' - x\,y.$$

Again the equation $y' - y = (x' - x) \,.\, P$ gives

$$a\,(y' - y) = (x' - x) \,.\, \frac{x\,y + a\,(y' + b)}{a}; \ldots .(a).$$

whence

$$a\,x'y' - a\,x\,y = x\,x'\,y - x^2\,y + a\,x'\,y' - a\,x\,y' + a\,b\,(x' - x),$$

that is,

$$a\,(y' - y) \,.\, x = (x\,y + a\,b)\,(x' - x).$$

This combined with the equation (a) gives

$$y' = \frac{(a - x)\,(x\,y + a\,b)}{a\,x},$$

whence it is easy to obtain

$$x' = \frac{a^2\,b}{x\,y + a\,b}.$$

These values give

$$P = \frac{x y + a b}{a x} = P',$$

which substituted in the equation

$$y = P x + f(P)$$

gives

$$y(a - x) = a b + a . f(P')$$

or simply changing the form of the function f,

$$y(a - x) = f\left(\frac{x y + a b}{a x}\right).$$

If we suppose the arbitrary function constant and equal to c^2, we get $y(a - x) = c^2$, the equation of an hyperbola.

SECTION XI.

On Circulating Equations.

(1). To find an analytical function of x, which when x is made to pass in succession through all integer values from 0 to infinity, shall assume in regular periodical rotation the n values $a, b, c \ldots . k, a, b, c \ldots . k,$ &c.

Let $\alpha, \beta, \gamma \ldots . \nu$ be the n^{th} roots of unity, and let

$$S_x = \frac{\alpha^x + \beta^x + \gamma^x \ldots . + \nu^x}{n},$$

then if we take

$$P_x = a . S_x + b . S_{x-1} + \ldots \ldots k . S_{x-n+1},$$

P_x will be the function required. The reason is obvious, when x is a multiple of n, the function S_x becomes unity by reason of the property of the roots of unity, demonstrated in works on Algebra, but in all other cases its value is zero.

Now some one of the values x, $x-1$, ... $x-n+1$, is necessarily such a multiple, and x being made to vary from 0 to infinity, this one will be either x, $x-1$, ... $x-n+1$, x, $x-1$, ... &c. in rotation, so that the function P_x will reduce itself to a, b, ... k, &c. in the same succession, and is therefore the function required.

(2). To find a function P_x which shall assume in regular periodical succession the same values as those of n other given functions a_x, b_x, c_x .... k_x. I say that

$$P_x = a_x . S_x + b_x . S_{x-1} + \dots . k_x . S_{x-n+1}.$$

For, the values of P_x corresponding to $x = 0, 1, 2, \dots . n-1, n, n+1, \dots$.&c. are respectively a_0, b_1, c_2, ... k_{n-1}, a_n, b_{n+1}, &c.

The functions described in the above paragraphs are "circulating functions," and may be distinguished into those with either constant or variable coefficients, of which we have here instances.

(3). *Theorem.* Any symmetrical function of S_x, S_{x-1}, ... S_{x-n+1} is invariable. For when x varies from 0 to ∞, some one of the values of these expressions is always unity and the rest zero, and, the function in question being symmetrical, it is no matter in what order this takes place, the order of its elements being of no consequence. The function therefore has the same value, whichever of its elements becomes unity, the rest being all zero. That is, it is invariable by the variation of x from integer to integer values. Thus

$$S_x + S_{x-1} + S_{x-2} + \dots . S_{x-n+1} = 1,$$

$$S_x . S_{x-1} . S_{x-2} \dots . S_{x-n+1} = 0.$$

(4). Every symmetrical function of the circulating functions P_x, P_{x-1}, ... P_{x-n+1} is in like manner invariable, provided the coefficients of P_x, &c. be constant.

For every such function is a symmetrical function of $S_x, S_{x-1}, \ldots S_{x-n+1}$, as will appear if we consider that by reason of the properties of the roots of unity, we have $S_{x-n} = S_x$, $S_{x-n-1} = S_{x-1}$, &c. and consequently,

$$P_x = a \cdot S_x + b \cdot S_{x-1} + \ldots k \cdot S_{x-n+1}$$

$$P_{x-1} = k \cdot S_x + a \cdot S_{x-1} + \ldots j \cdot S_{x-n+1}$$

$$P_{x-2} = j \cdot S_x + k \cdot S_{x-1} + \ldots i \cdot S_{x-n+1}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$P_{x-n+1} = b \cdot S_x + c \cdot S_{x-1} + \ldots a \cdot S_{x-n+1}.$$

Now any symmetrical function of the second members of these equations will obviously involve $S_x, S_{x-1}, \ldots S_{x-n+1}$, symmetrically, and will therefore be invariable. Its value also will evidently be equal to that of a function similarly composed of the coefficients a, b, &c. Thus for instance, if $P_x = a \cdot S_x + b \cdot S_{x-1}$, ($n$ being $= 2$) we have,

$$P_x \cdot P_{x-1} = (a \cdot S_x + b\, S_{x-1})\,(a \cdot S_{x-1} + b \cdot S_x)$$

$$= a\, b\, (S_x)^2 + a\, b\, (S_{x-1})^2 = a\, b\, (S_x^2 + S_{x-1}^2) = a\, b.$$

(5). For instances we may take

$$P_x \cdot P_{x-1} \ldots P_{x-n+1} = a \cdot b \cdot c \ldots k\,;$$

$$P_x + P_{x-1} + \ldots P_{x-n+1} = a + b + c + \ldots k\,; \text{ \&c.}$$

(6). Circulating equations are those whose coefficients are circulating functions. To resolve them they must be reduced to others, whose coefficients are of the ordinary form. The preceding propositions enable us to do this. To begin with a simple instance, let

$$u_{x+2} \pm P_x \cdot u_{x+1} \pm u_x = 0,$$

where P_x is a circulating function *of the second degree* (or in which $n = 2$, $P_x = a \cdot S_x + b \cdot S_{x-1}$).

Assume $u_x = v_x \,.\, \sqrt{(P_x)}$,

then will $u_{x+2} = v_{x+2} \sqrt{(P_{x+2})} = v_{x+2} \,.\, \sqrt{(P_x)}$, and the equation becomes

$$v_{x+2} \sqrt{(P_x)} \pm P_x \,.\, v_{x+1} \,.\, \sqrt{(P_{x+1})} \pm v_x \sqrt{P_x} = 0,$$

or

$$v_{x+2} \pm v_{x+1} \,.\, \sqrt{(P_x \,.\, P_{x+1})} \pm v_x = 0.$$

Now the coefficient of the second term $\sqrt{(P_x P_{x+1})}$ being a symmetrical function, is invariable by (4) and equal to $\sqrt{(a\, b)}$. We have, therefore,

$$v_{x+2} \pm \sqrt{(a\, b)} \cdot v_{x+1} \pm v_x = 0,$$

an ordinary equation with constant coefficients, and easily integrated.

(7). A more general process however, and applicable to all circulating equations is to assume for the independent variable, a circulating function with unknown and variable coefficients, as in the following equation,

$$u_x + (a \,.\, S_x + b \,.\, S_{x-1})\, u_{x-1} + (\alpha \,.\, S_x + \beta \,.\, S_{x-1}) = 0.$$

Assume $u_x = A_x \,.\, S_x + B_x \,.\, S_{x-1}$, and we have by substitution and by Art. 5. of this Section,

$$\left.\begin{array}{ll} A_x \,.\, S_x & + \; B_x \,.\, S_{x-1} \\ + \; a\, B_{x-1}\, S_x & + \; b \,.\, A_{x-1}\, S_{x-1} \\ + \; \alpha \,.\, S_x & + \; \beta \,.\, S_{x-1} \end{array}\right\} = 0;$$

whence, equating to zero the coefficients of S_x and S_{x-1} separately, we obtain

$$A_x + a \,.\, B_{x-1} + \alpha = 0; \qquad B_x + b \,.\, A_{x-1} + \beta = 0.$$

Eliminating B_x, we find

$$A_x - a\, b \,.\, A_{x-2} + (\alpha - a\, \beta) = 0,$$

whence, A_x is found, equal to

$$(C . S_x + C' . S_{x-1}) \{ \surd(ab) \}^x - \frac{\alpha - a\beta}{1 - ab},$$

and thence we derive by the second of the above equations,

$$B_x = - b \{ \surd(ab) \}^{x-1} (C' . S_x + C . S_{x-1}) - \frac{\alpha - a\beta}{1 - ab}.$$

If these be substituted in the expression for u_x, we get

$$u_x = C \{ \surd(ab) \}^x . \{ \surd a . S_x - \surd b . S_{x-1} \}$$
$$- \frac{(\alpha - a\beta) S_x + (\beta - b\alpha) S_{x-1}}{1 - ab},$$

which contains (as it ought) only one arbitrary constant C.

(8). Suppose the equation were

$$u_x + (a . S_x + b . S_{x-1}) u_{x-1} + c = 0.$$

Here $c = c . S_x + c . S_{x-1}$; therefore this is only a particular case of the preceding, and so of any other constant coefficient in a circulating equation. This gives, consequently,

$$u_x = C \{ \surd(ab) \}^x . \{ \surd a . S_x - \surd b . S_{x-1} \}$$
$$- c . \frac{(1 - a) S_x + (1 - b) S_{x-1}}{1 - ab}.$$

(9). Let the proposed equation be

$$u_{x+1} - R . u_x = P_x,$$

where R is constant, and P_x any circulating function the period of circulation being n, or

$$P_x = a . S_x + b . S_{x-1} + \dots . k . S_{x-n+1}.$$

(10). Let the equation to be integrated be

$$u_{x+1} - R . u_x + P_x = 0.$$

where R is constant, and P_x any circulating function of the form

$$a\,S_x + b\,S_{x-1} + c\,S_{x-2} + \ldots . k\,S_{x-n+1}.$$

Assume

$$u_x = A_x . S_x + B_x . S_{x-1} + \ldots . K_x . S_{x-n+1},$$

then

$$u_{x+1} = B_{x+1}\,S_x + C_{x+1}\,S_{x-1} + \ldots . A_{x+1}\,S_{x-n+1},$$

and the equation becomes by substitution

$$0 = (B_{x+1} - R\,A_x + a)\,S_x + (C_{x+1} - R\,.\,B_x + b)\,S_{x-1} + \ldots\ldots$$
$$\ldots\ldots\ldots\ldots + (A_{x+1} - R\,.\,K_x + k)\,S_{x-n+1}$$

whose terms severally equated to zero give

$$B_{x+1} = R\,A_x - a,$$
$$C_{x+1} = R\,B_x - b,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$A_{x+1} = R\,K_x - k,$$

whence we get

$$C_{x+2} = R^2\,A_x - (R\,a + b),$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$A_{x+n} = R^n\,A_x - (a\,R^{n-1} + b\,R^{n-2} + \ldots\ldots k).$$

This last equation integrated gives

$$A^x = R^x . \{ C . S_x + C' . S_{x-1} + \ldots\ldots C . S_{x-n+1} \}$$
$$- \frac{a\,R^{n-1} + b\,R^{n-2} \ldots . + k}{1 - R^n},$$

C, C', ... being n arbitrary constants. Now since we have

$$u_x = A_x . S_x + B_x . S_{x-1} + \&c.,$$

we may neglect in the value of A_x all the terms but that multiplied by S_x, in B_x all but that multiplied by S_{x-1} &c.

which comes ultimately to the same as making C', C'', &c. $= 0$, because u_x can only contain one arbitrary constant C*. This done we get (putting Q for the constant part of the value of A_x),

$$A_x = C \, . \, R^x S_x - Q,$$

$$B_x = C \, . \, R^x S_{x-1} - (R\,Q + a),$$

. . . .

$$K_x = C \, . \, R^x S_{x-n+1} - (R^{n-1}Q + a\,R^{n-2} + b\,R^{n-3} \ldots . + j)$$

which substituted give

$$u_x = C \, . \, R^x - \frac{a\,R^{n-1} + b\,R^{n-2} + \ldots . k}{1 - R^n} \, . \, S_x,$$

$$- \frac{b\,R^{n-1} + \ldots . k\,R + a}{1 - R^n} S_{x-1},$$

. . . .

$$- \frac{k\,R^{n-1} + a\,R^{n-2} + \ldots . j}{1 - R^n} S_{x-n+1}.$$

(11). Let the equation be

$$u_{x+1} - P_x u_x + P'_x = 0,$$

P_x and P'_x being respectively equal to

$$a\,S_x + b \, . \, S_{x-1} + \ldots . k \, . \, S_{x-n+1},$$

and

$$\alpha\,S_x + \beta\,S_{x-1} + \ldots . \kappa\,S_{x-n+1}.$$

Making the same substitution for u_x, the equations for determining A_x, &c. will be

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* The same result will be obtained, if we retain all the constants and investigate in general the values of A_x, B_x,......K_x. If these be then substituted in the expression for u_x, the supernumerary constants will all destroy each other, of which we have already seen an instance in Number 7, of this Section.

$$B_{x+1} = a \, . \, A_x - \alpha$$
$$C_{x+1} = b \, . \, B_x - \beta,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$K_{x+1} = j \, . \, J_x - i$$
$$A_{x+1} = k \, . \, K_x - \kappa.$$

These give

$$C_{x+2} = a\,b\,A_x - (\alpha\,b + \beta)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$K_{x+n-1} = a\,b\,c \ldots . j \, . \, A_x - (b\,c \ldots . j \, . \, \alpha + c \ldots . j \, . \, \beta + \ldots i)$$
$$A_{x+n} = a\,b\,c \ldots . k \left\{ A_x - \frac{\alpha}{a} - \frac{\beta}{a\,b} - \ldots . \frac{\kappa}{a\,b \ldots . k} \right\},$$

which integrated gives

$$A_x = (a\,b\,c \ldots . k)^{\frac{x}{n}} \, . \, \{ C \, . \, S_x + C' \, . \, S_{x-1} + \&c. \}$$
$$- \frac{\frac{\alpha}{a} + \frac{\beta}{a\,b} + \ldots . \frac{\kappa}{a\,b \ldots . k}}{1 - a\,b\,c \ldots . k} \, . \, (a\,b\,c \ldots . k)$$

and putting Q for the constant part, and N for $(a \, . \, b \ldots . k)^{\frac{1}{n}}$ and, effacing all the arbitrary constants but the first, as in the last number,

$$A_x = C\,N^x \, . \, S_x - Q,$$
$$B_x = C \, . \, N^{x-1} \, S_{x-1} - (a\,Q + \alpha),$$
$$C_x = C \, . \, a\,b\,N^{x-2} \, S_{x-2} - (a\,b \, . \, Q + b\,\alpha + \beta)\,; \&c.$$

which substituted give

$$u_x = C \, . \, \{ N^x \, . \, S_x + a\,N^{x-1} \, S_{x-1} + \ldots \ldots$$
$$\ldots (a\,b\,c \ldots . j)\,N^{x-n+1} \, S_{x-n+1} \},$$
$$- \frac{N}{1 - N} \left(\frac{\alpha}{a} + \frac{\beta}{a\,b} + \ldots . \frac{\kappa}{a\,b \ldots . k} \right) S_x,$$

$$-\frac{N}{1-N}\left(\frac{\beta}{b}+\frac{\gamma}{bc}+\ldots\frac{a}{bc\ldots ka}\right)S_{x-1},$$

. . . .

$$-\frac{N}{1-N}\left(\frac{\kappa}{k}+\frac{a}{ka}+\ldots\frac{\nu}{kab\ldots j}\right)S_{x-n+1}.$$

(12). It is sometimes necessary to reduce two or more circulating functions with different periods of circulation to a common period. This is easily accomplished; suppose for instance the functions were

$$a\,.\,S_x+b\,.\,S_{x-1},\quad \text{(where } n=2\text{)},$$

and

$$\alpha\,.\,S_x+\beta\,.\,S_{x-1}+\gamma\,.\,S_{x-2},\quad \text{(where } n=3\text{)}.$$

To denote these and similar functions more readily, suppose we take

$$S_x^{(n)}=\frac{\text{sum of } x^{\text{th}} \text{ powers of } n^{\text{th}} \text{ roots of unity}}{n}.$$

Then will the two functions in question be represented by

$$a\,.\,S_x^{(2)}+b\,.\,S_{x-1}^{(2)},$$
$$\alpha\,.\,S_x^{(3)}+\beta\,.\,S_{x-1}^{(3)}+\gamma\,.\,S_{x-2}^{(3)}.$$

The first of these is obviously equal to

$$a\,.\,S_x^{(6)}+b\,.\,S_{x-1}^{(6)}+a\,.\,S_{x-2}^{(6)}+b\,.\,S_{x-3}^{(6)}$$
$$+\,a\,.\,S_{x-4}^{(6)}+b\,.\,S_{x-5}^{(6)}.$$

Because when either x, or $x-2$, or $x-4$, is a multiple of 6, it is so of 2, and therefore x is a multiple of 2, consequently both functions reduce themselves to a; again, if $x-1$, or $x-3$, or $x-5$, be a multiple of 6, and therefore of 2, $x-1$ must necessarily be a multiple of 2, so that both reduce themselves to b. In like manner it may be proved that the other given function is identical with

$$\alpha . S_x^{(6)} + \beta . S_{x-1}^{(6)} + \gamma . S_{x-2}^{(6)}$$
$$+ \alpha . S_{x-3}^{(6)} + \beta . S_{x-4}^{(6)} + \gamma . S_{x-5}^{(6)},$$

so that the two are thus reduced to the common period $6 = 2 \times 3$.

If the separate periods have a common measure, the compound period will be their product, divided by this common measure. The reason is obvious.

By this means should equations occur involving circulating functions with different periods, they may be integrated.

(13). The following general property of circulating functions may be mentioned in addition to those enumerated in 3, 4, 5.

Let

$$P_x = a_x . S_x + b_x . S_{x-1} + \ldots . k_x . S_{x-n+1},$$

Then

$$f(P_x) = f(a_x) . S_x + f(b_x) . S_{x-1} + \ldots . f(k_x) . S_{x-n+1},$$

or *any function of a circulating function is itself a circulating function, whose coefficients are similar functions of those of the original one respectively.* In like manner, if P'_x, P''_x, &c. be other circulating functions,

$$f(P_x, P'_x, \ldots) = f(a_x, a'_x, \ldots) . S_x + f(b_x, b'_x, \ldots) S_{x-1} + \&c.$$

Thus for instance (if the coefficients be constant)

$$\frac{P_x + P'_x}{1 - P_x P'_x} = \frac{a + a'}{1 - a a'} S_x + \frac{b + b'}{1 - b b'} S_{x-1} + \&c.$$

we shall have occasion to recal this principle hereafter. It is too evident to require a formal demonstration.

(14). To determine the integrals ΣS_x and ΣP_x.

Putting $\Sigma S_x = u_x$, we have $u_{x+1} - u_x = S_x$, and this may be treated as a circulating equation; thus assuming

$$u_x = A_x \,.\, S_x + B_x \,.\, S_{x-1} + \ldots . K_x \,.\, S_{x-n+1},$$

we have

$$0 = (B_{x+1} - A_x - 1)\,S_x + (C_{x+1} - B_x)\,S_{x-1} + \ldots .$$

$$\ldots\ldots (A_{x+1} - K_x)\,S_{x-n+1},$$

whose terms severally equated to zero give

$$B_{x+1} = 1 + A_x$$

$$C_{x+1} = B_x$$

. . . .

$$A_{x+1} = K_x,$$

whence we get

$$A_{x+n} = 1 + A_x.$$

A particular solution of this will suffice for our purpose, and it is evident that $A_x = \frac{x}{n}$ will satisfy it. This gives

$$B_x = \frac{x-1}{n} + 1\,; \quad C_x = \frac{x-2}{n} + 1, \ldots . \&c.$$

and finally

$$u_x = \frac{1}{n}\,\{x \,.\, S_x + (x+n-1)\,S_{x-1} + (x+n-2)\,S_{x-2} \ldots .$$

$$\ldots . (x+1)\,S_{x-n+1}\} + \text{const.}$$

$$= \text{const} + \frac{x}{n}(S_x + S_{x-1} + \ldots . S_{x-n+1})$$

$$+ \frac{(n-1)\,S_{x-1} + \ldots . 1 \,.\, S_{x-n+1}}{n}$$

$$= \text{const} + \frac{x}{n} + \frac{1}{n}\,\{(n-1)\,S_{x-1} + (n-2)\,S_{x-2} \ldots .$$

$$\ldots . + 1 \,.\, S_{x-n+1}\}\,.$$

Hence, if $P_x = a \,.\, S_x + b \,.\, S_{x-1} + \ldots . k \,.\, S_{x-n+1}$, we get

$$\Sigma P_x = \text{const} + (a + b + \ldots . k) \,.\, \frac{x}{n}$$

$$+ \frac{1}{n} \{ 0 \,.\, a + 1 \,.\, b + \ldots . (n-1) \,.\, k \} \, S_x$$

$$+ \frac{1}{n} \{ (n-1) \,.\, a + 0 \,.\, b + \ldots . (n-2) \,.\, k \} \, S_{x-1}$$

$$+ \frac{1}{n} \{ (n-2) \,.\, a + (n-1) \,.\, b + 0 \,.\, c + \ldots . \} \, S_{x-2}$$

. . . .

$$+ \frac{1}{n} \{ 1 \,.\, a + 2 \,.\, b + \ldots . (n-1) \,.\, j + 0 \,.\, k \} \, S_{x-n+1}.$$

SECTION XII.

Of continued Fractions.

(1). To determine the value of the continued fraction,

$$\frac{c_1}{a_1 + \frac{c_2}{a_2 + \frac{c_3}{a_3 + \ldots \ldots \frac{c_x}{a_x}}}},$$

or, as it may more conveniently be written *,

$$\frac{c_1}{a_1 +} \; \frac{c_2}{a_2 +} \; \frac{c_3}{a_3 +} \ldots \ldots \frac{c_x}{a_x}.$$

Let the fraction be put equal to u_x. Then we have

$$u_1 = \frac{c_1}{a_1}, \quad u_2 = \frac{c_1}{a_1 + \frac{c_2}{a_2}} = \frac{c_1 \,.\, a_2}{a_1 \,.\, a_2 + c_2},$$

* After the example of Burmann.

and so on. It appears then, that u_x is reducible to the form $\frac{N_x}{D_x}$, N_x and D_x being the numerator and denominator of a certain rational fraction, each composed of combinations of a_1, a_2, &c. c_1, c_2, &c. formed by multiplication and addition. Let us now examine them more closely. To this effect we have

$$u_3 = \frac{c_1}{a_1 + \frac{c_2 a_3}{a_2 a_3 + c_3}} = \frac{c_1 a_2 . a_3 + c_1 . c_3}{(a_1 a_2 + c_2) a_3 + a_1 . c_3}$$

$$u_4 = \frac{c_1}{a_1 + \frac{c_2 a_3 . a_4 + c_2 . c_4}{(a_2 a_3 + c_3) a_4 + a_2 . c_4}}$$

$$= \frac{(c_1 a_2 . a_3 + c_1 . c_3) a_4 + c_1 a_2 . c_4}{(a_1 a_2 a_3 + a_1 c_3 + c_2 a_3) a_4 + (a_1 a_2 + c_2) c_4}$$

and so on. Hence, we have the following series of equations:

$$N_1 = c_1$$
$$N_2 = c_1 . a_2$$
$$N_3 = c_1 . c_3 + c_1 a_2 . a_3$$
$$N_4 = c_1 a_2 . c_4 + (c_1 . c_3 + c_1 a_2 . a_3) a_4 ; \text{ \&c.}$$

that is,

$$N_1 = N_1$$
$$N_2 = N_2$$
$$N_3 = c_3 . N_1 + a_3 . N_2$$
$$N_4 = c_4 . N_2 + a_4 . N_3$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$N_{x+2} = c_{x+2} . N_x + a_{x+2} . N_{x+1} ; \quad (1).$$

Similarly, for the denominators, we have

$$D_1 = a_1, \quad D_2 = c_2 + a_1 a_2$$
$$D_3 = a_1 . c_3 + (c_2 + a_1 a_2) . a_3$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$D_{x+2} = a_{x+2} D_{x+1} + c_{x+2} D_x ; \quad (2).$$

The integration of the equations (1) and (2) will therefore lead to the values of N_x and D_x, and therefore to that of their quotient u_x. As these two equations are precisely the same, the complete integral of one is also that of the other, and the values of N_x and D_x of course can only differ by reason of the different values of the arbitrary constants which enter into their expressions.

The equations (1) and (2) were noticed at the first origin of the theory of continued fractions, by Wallis in his Arithmetica Infinitorum, Prop. 191. p. 192. (Opera Wallisii. Oxon. 1657.) as rules for the ready computation of the approximating limits of infinite fractions of this kind, for which purpose they are well adapted, as they enable us to deduce the successive numerators and denominators of the limiting vulgar fractions $\frac{N_1}{D_1}$, $\frac{N_2}{D_2}$, $\frac{N_3}{D_3}$, &c. one from the other very readily.

(2).. Required the value of the continued fraction

$$\frac{c}{a+}\,\frac{c}{a+}\,\frac{c}{a+}\ldots\ldots \text{ (to } x \text{ terms).}$$

Here c_x and a_x being constant, the integral of the equation

$$N_{x+2} = a \, . \, N_{x+1} + c\, N_x,$$

is $N_x = C \, . \, \alpha^x + C' \, . \, \beta^x$, α and β being the roots of $z^2 = az + c$. This integral may be expressed more conveniently for the present purpose by changing the arbitrary constants C and C' into $C + \frac{c'}{\alpha}$, and $C + \frac{c'}{\beta}$ which does not diminish the generality of the equation*, but only reduces it to the more symmetrical form

$$N_x = C(\alpha^x + \beta^x) + c'(\alpha^{x-1} + \beta^{x-1}).$$

* This change of the arbitrary constants might have been made with advantage in Art. 393. Appendix, where it would have dispensed with a good deal of pretty abstract reasoning, but it did not occur at the time.

Determining the constants then so that $N_1 = c$, $N_2 = a\,c$, we get

$$N_x = \frac{c}{a^2 + 4c}\left\{ a(\alpha^x + \beta^x) + 2c(\alpha^{x-1} + \beta^{x-1}) \right\},$$

and since $D_1 = a$, and $D_2 = a^2 + c$, we find in like manner

$$D_x = \frac{1}{a^2 + 4c}\left\{ (a^2 + 2c)(\alpha^x + \beta^x) + a\,c(\alpha^{x-1} + \beta^{x-1}) \right\}.$$

So that the general expression for u_x is

$$u_x = \frac{a\,c(\alpha^x + \beta^x) + 2\,c^2(\alpha^{x-1} + \beta^{x-1})}{(a^2 + 2c)(\alpha^x + \beta^x) + a\,c(\alpha^{x-1} + \beta^{x-1})}.$$

Let U represent the fraction continued to infinity, and suppose α the greater of the roots of the equation $z^2 - a\,z - c = 0$, (without regard to its sign,) and we get by making x infinite

$$U = \frac{a\,c\,\alpha + 2c^2}{(a^2 + 2c)\,\alpha + a\,c},$$

which, (as may easily be proved,) is one of the roots of the quadratic

$$U^2 + a\,U - c = 0.$$

Thus

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+\text{\&c.}}\ \textit{ad inf.} = \frac{\sqrt{5} - 1}{2},$$

$$\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+\text{\&c.}}\ \textit{ad inf.} = \sqrt{2} - 1,$$

$$\frac{2}{1+}\,\frac{2}{1+}\,\ldots\ldots(\text{to } x \text{ terms}) = \frac{2^{x+1} + 2(-1)^{x+1}}{2^{x+1} + (-1)^x}.$$

and to infinity $= 1$.

(3). Required the value of the continued fraction

$$u_x = \frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\&c.$$

continued to x terms, that is, containing x fractional terms.

The period of the denominators being a, b, we may assume $c_x = 1$, $a_x = b\,S_x + a\,S_{x-1}$, where S_x is the sum of the x^{th} powers of the roots of $z^2 - 1 = 0$, and we have by (Art. 1.)

$$N_{x+2} - N_x = (b\,S_x + a\,S_{x-1})\,N_{x+1}.$$

This equation is the same with that integrated in 6, Sect. 11. and taking

$$N_x = v_x \,.\, \sqrt{(b\,.\,S_x + a\,.\,S_{x-1})}$$
$$= v_x \,.\, (\sqrt{b}\,.\,S_x + \sqrt{a}\,.\,S_{x-1}),\quad \text{by 12, Sect. 11.}$$

we have seen that v_x is given by the equation

$$v_{x+2} - \sqrt{(a\,b)}\,.\,v_{x+1} - v_x = 0.$$

Let then α, β, be the two roots of

$$z^2 - \sqrt{(a\,b)}\,.\,z - 1 = 0,$$

and we have

$$N_x = \{\sqrt{a}\,.\,S_{x-1} + \sqrt{b}\,.\,S_x\}\,[c\,(\alpha^x + \beta^x) + c'\,(\alpha^{x-1} + \beta^{x-1})].$$

To determine the constants we have $N_1 = 1$, $N_2 = b$, and since the value of D_x is precisely similar, the constants only being determined by making $D_1 = a$, $D_2 = a\,b + 1$, we obtain after substitution and all reductions the following value of the continued fraction

$$\frac{N_x}{D_x} = \sqrt{\frac{b}{a}}\,.\,\frac{\sqrt{(a\,b)}\,(\alpha^x + \beta^x) + 2\,(\alpha^{x-1} + \beta^{x-1})}{(a\,b+2)\,(\alpha^x + \beta^x) + \sqrt{(a\,b)}\,(\alpha^{x-1} + \beta^{x-1})}.$$

As before, let α be the greatest root of the equation $z^2 - \sqrt{(a\,b)}\,.\,z - 1 = 0$, without regard to its sign, and let U

represent the value of the fraction ad infinitum. Then by making x infinite, we get

$$U = \sqrt{\frac{b}{a}} \cdot \frac{2 + a \cdot \sqrt{(a b)}}{\sqrt{(a b)} + a (a b + 2)},$$

which is readily shewn to be a root of the equation

$$U^2 + b U = \frac{b}{a},$$

by substituting for $\sqrt{(a b)}$ its value $a - \frac{1}{a}$.

(4). To determine the value of the fraction

$$1 + \frac{c_x}{1 +} \frac{c_{x-1}}{1 +} \cdots\cdots \frac{c_1}{1},$$

regarded as a function of x.

If we would employ the preceding investigations we must regard x as constant, and assume another independent variable z, and another function of it p_z, such that

$$c_z = p_{x-z+1}.$$

This gives

$$c_x = p_1, \quad c_{x-1} = p_2, \cdots\cdots c_{x-z+1} = p_z.$$

If then we enquire by the methods above delivered the value of the expression

$$1 + \frac{p_1}{1 +} \frac{p_2}{1 +} \cdots\cdots \frac{p_z}{1}$$

in its general form, as a function of z, and then write in the result for p_z and its derivative functions, the functions c_{x-z+1} and others similarly derived from it *as a function of* z, and finally put $z = 1$, we have the value required. But the following process is simpler, and less liable to mistake.

Assume u_x for the value required. Then

$$u_{x+1} = 1 + \frac{c_{x+1}}{u_x}.$$

$$u_{x+1} u_x = u_x + c_{x+1},$$

or, taking

$$u_x = \frac{v_{x+1}}{v_x}$$

$$v_{x+2} = v_{x+1} + c_{x+1} v_x$$

an equation of differences of the second order, the form of which it will be remarked is precisely that which determines N_{x-1} and D_{x-1} in the value $\frac{N_{x-1}}{D_{x-1}}$ of the function

$$\frac{c_1}{1+} \frac{c_2}{1+} \cdots\cdots \frac{c_{x-1}}{1}. \quad \text{(See Prob. 1.)}$$

But though the equation of differences is the same, the nature of the functions derived from it will be essentially modified by the different constants required to adapt its integral to the two cases. Still, however, this coincidence assigns a relation between the two functions sufficiently remarkable.

In fact, let $(A_x + C \,.\, B_x) \times C'$ be the general value of v_x. Then will

$$u_x - 1 = \frac{\Delta A_x + C \,.\, \Delta B_x}{A_x + C \,.\, B_x},$$

and

$$\frac{N_{x-1}}{D_{x-1}} = C' \,.\, \frac{A_x + C'' B_x}{A_x + C'' B_x}$$

where the constants depend on the values of c_1, c_2, A_1, A_2, B_1, B_2, and are easily determined. Now these expressions are respectively the values of

$$\frac{c_x}{1+}\ \frac{c_{x-1}}{1+}\ \ldots\ldots\ \frac{c_1}{1}$$

and

$$\frac{c_1}{1+}\ \frac{c_2}{1+}\ \ldots\ldots\ \frac{c_x}{1}.$$

(5). Having given the value of

$$1+\frac{c_x}{1+}\ \frac{c_{x-1}}{1+}\ \ldots\ldots\ \frac{c_1}{1}=V_x.$$

To determine that of the same fraction with any additional denominator at the end,

$$1+\frac{c_x}{1+}\ \frac{c_{x-1}}{1+}\ \ldots\ldots\ \frac{c_1}{a}.$$

If we proceed as above, by putting $\frac{v_{x+1}}{v_x}$ for its value we shall obtain the very same equation of differences and therefore the expressions of the two functions can only differ in the values of the arbitrary constant C in the expression $\frac{A_{x+1}+C\,.\,B_{x+1}}{A_x+C\,.\,B_x}$. Now since one value V_x of $\frac{v_{x+1}}{v_x}$ is given, one value of v_x is also known, being equal to $V_1\,.\,V_2\ldots\ldots V_{x-1}$. Hence a particular integral of the equation for v_x is given, and of course, being only of the second order, the complete integral may be ascertained by the methods delivered in the text (Append. Art. 382.): and the constant must then be adapted to the case in question.

(6). To find, for instance, the value u_x of the fraction

$$1+\frac{c}{1+}\ \frac{c}{1+}\ \ldots\ldots\ \frac{c}{a},$$

when the letter c occurs x times

$$u_x = \frac{a^{x+1} + C.\beta^{x+1}}{a^x + C.\beta^x}; \quad C = -\frac{a\left(1 + \frac{c}{a}\right) - a^2}{\beta\left(1 + \frac{c}{a}\right) - \beta^2}$$

a and β being the roots of $v^2 - v - c = 0$.

SECTION XIII.

Application of the Calculus of Differences to various Problems.

(1). WHAT are the respective amounts of a given sum for x years, at simple, and at compound interest?

Let P_x be the amount at the end of the x^{th} year, then 1st, at simple interest, if A be the original sum, and r the interest of £.1 for 1 year, rA will be that of £A, and therefore the increase in one year being rA, $P_x + rA$ is the amount at the end of the $(x+1)^{th}$ year, but this amount is also represented by P_{x+1}. Hence

$$P_{x+1} = P_x + rA, \text{ or } \Delta P_x = rA,$$

and integrating

$$P_x = rA.x + C.$$

Now the original sum or value of P_x, when $x = 0$, is A, hence

$$P_0 = rA.0 + C = A, \text{ or } C = A,$$

and therefore,

$$P_x = A(1 + rx).$$

2d, At compound interest. P_x being the capital at the end of the x^{th} year, $r P_x$ is the interest in the $(x+1)^{th}$ therefore,

$$P_{x+1} = P_x + r P_x = (1+r) \cdot P_x,$$

and integrating,

$$P_x = C(1+r)^x.$$

Now $P_0 = A$, hence $C = A$, and $P_x = A(1+r)^x$.

(2). A person places money in the funds, but gradually contracting expensive habits, he spends the first year the whole interest, the second twice that of the remaining stock, the third three times that of what is left, and so on. How long will his property last, and in what year is his expenditure greatest?

As before, let P_x=his stock at the end of the x^{th} year, r=interest of £1, for 1 year.

Then $r P_x$ = that upon P_x, and consequently his expenditure in the $(x+1)^{th}$ year is $(x+1) r P_x$. Therefore at the end of the $(x+1)^{th}$ year his stock will be

$$P_x + r P_x - (x+1) r P_x = P_x (1 - x r).$$

Hence,

$$P_{x+1} = (1 - x r) P_x,$$

and integrating on the hypothesis $P_0 = A$,

$$P_x = A \cdot 1 (1-r)(1-2r) \ldots \ldots \{1 - (x-1) \cdot r\}.$$

This vanishes when $x = 1 + \frac{1}{r}$, which is the number of years his stock will last. Also, his expenditure in the x^{th} year being $x r P_{x-1}$, and in the next, $(x+1) r P_x$ will be greatest just before $\Delta (x r P_{x-1})$ becomes negative, because then having reached its maximum it begins to decrease.

Now

$$\Delta (x r P_{x-1}) = 1 . (1 - r) \ldots\ldots$$
$$\ldots\ldots \{ 1 - (x - 1). r \} \; \{ (x + 1)(1 - x r) - x \} .$$

Suppose then

$$(x + 1)(1 - x r) - x = 0,$$

this gives

$$x = \sqrt{\left(\frac{1}{r} + \frac{1}{4}\right)} - \frac{1}{2},$$

and the nearest integer less than this is the required number. If $r = \frac{1}{20}$, $x = 4$ exactly. Here then $\Delta (x P_{x-1}) = 0$, when $x = 4$, so that the sums spent in the 4th and 5th years are equal, and greater than in any other.

(3). A person puts out to interest a sum of money (A), he expends annually a portion (a) of the interest, and adds the remainder to the stock. What is the amount after x years?

Call it P_x; then the interest is $r P_x$, so that

$$P_{x+1} = P_x + r P_x - a, \text{ or}$$
$$P_{x+1} - (1 + r) P_x + a = 0.$$

which integrated gives

$$P_x = C (1 + r)^x + \frac{a}{r}.$$

The constant C, must be determined by the consideration that P_0 the original stock is equal to A which gives

$$C = A - \frac{a}{r},$$

and

$$P_x = \left(A - \frac{a}{r}\right)(1 + r)^x + \frac{a}{r}.$$

If a exceed the interest, and on that supposition we would find how long the money will last, make $P_x = 0$, and we get

$$x = \frac{\log a - \log (a - r A)}{\log (1 + r)}.$$

If we would find what annual sum a stock-holder may expend so that his property shall just last out his life, on a fair calculation, call x the number of years he has a reasonable expectation of living (calculated from the tables of mortality) then we have

$$a = r A \cdot \frac{(1 + r)^x}{(1 + r)^x - 1}.$$

(4). A has £.1000 ($= A$) in the funds at 5 per cent. ($= r$). He spends the first year the full interest of his capital £.500, the next £.1000, and so on in regular arithmetical progression. How long will his property last?

As before, putting P_x for his property at the end of the x^{th} year, we have

$$P_{x+1} = P_x (1 + r) - (x + 1) \cdot r A,$$

which integrated gives

$$P_x = A \left\{ (x + 1) - \frac{(1 + r)^x - 1}{r} \right\}.$$

Hence we have to find x from the exponential equation,

$$(1 + r)^x = 1 + r + r x,$$

and since $r = \frac{1}{20}$, $x = 6 . 6$ nearly.

Were a more exact value required, we must proceed thus: Suppose $f(x) = 0$, and a being an approximate value of x, let $a + h$ be the true value, then will h be small, and the equation

$$f(a + h) = 0,$$

developed by Taylor's theorem gives, neglecting h^2, &c.

$$f(a) + h \cdot \frac{df(a)}{da} = 0,$$

whence,

$$h = -\frac{f(a) \cdot da}{df(a)}, \quad \text{and } x = a + h,$$

is a second approximation.

In the present case,

$$f(a) = (1 + r)^a - 1 - r - ra,$$

$$\frac{df(a)}{da} = (1 + r)^a \cdot \log(1 + r) - r,$$

and

$$h = \frac{(1 + r + ra) - (1 + r)^a}{(1 + r)^a \log(1 + r) - r},$$

$$= -0.00453,$$

so that a second approximation is $x = 6.59547$.

(5). A man spends every year twice the sum he gained by a certain business the year before. That business, however, becomes every year more and more profitable, and he finds his property increase regularly, as the square of the time since he began business. In what progression do the profits of his trade increase?

If we call P_x the profit in the x^{th} year, the problem leads to the equation

$$P_{x+1} - 2P_x = a(2x + 1),$$

whence we find

$$P_x = a\{5 \cdot 2^{x-1} - (2x + 3)\} + P_1 \cdot 2^{x-1}.$$

Now $P_1 = a \cdot 1^2$, hence $P_x = a(3 \cdot 2^x - 2x - 3)$.

(6). An individual sets out in the world with a certain capital (A), one-half of which he places in the funds at r per cent. and the rest, ventures in a concern which produces

$2r$ per cent. per annum, but which returns the interest only once in two years; he lives at a stated rate of expenditure (a £ per annum) and puts all his gains and savings into the funds. Required his funded property after any number (x) of years.

Call his funded property at the end of the (x^{th}) year P_x. Then, if x be an even number, the interest on that part of his stock $\left(\frac{A}{2}\right)$ which is vested in trade does not accrue in the $(x+1)^{th}$ year, so that, (x being even),

$$P_{x+1} = (1+r)P_x - a,$$

but when x is odd the interest for two years at $2r$ per annum, accrues on the capital $\frac{A}{2}$, that is, $4r\frac{A}{2}$ or $2rA$, hence, (x being odd),

$$P_{x+1} = (1+r)P_x - a + 2rA.$$

It may not, perhaps, immediately appear how these equations are to be treated; because in either of them, if x be increased by unity, the equation ceases to be true, and therefore the function P_x cannot be found by integrating either of them separately, the law of continuity being broken. To supply this, and to include the odd and even values of x in one analysis, we must have recourse to the theory of circulating functions above delivered. In fact, since the circulating function of the second degree

$$a \,.\, S_x + (a - 2rA) \,.\, S_{x-1},$$

is equal either to a, or to $a - 2rA$, according as x is even or odd, the equation

$$P_{x+1} - R \,.\, P_x + \{ a \,.\, S_x + (a - 2rA) S_{x-1} \} = 0,$$

(where $R = 1 + r$) includes both the others, and admits of any value being assigned to x. The law of continuity being thus restored we may find P_x by integration, and we get (9, Sect. 11.)

$$P_x = C \,.\, R^x - \frac{a R + a - 2 r A}{1 - R^2} S_x$$

$$- \frac{(a - 2 r A) R + a}{1 - R^2} S_{x-1},$$

and determining C so that $P_0 = \frac{1}{2} A$, his funded property at the outset, and writing for S_x and S_{x-1} their values

$\frac{1 + (-1)^x}{2}$ and $\frac{1 - (-1)^x}{2}$, we get finally,

$$P_x = A \left\{ \frac{(r+6)(1+r)^x + 2 r (-1)^x}{2 (r+2)} - 1 \right\} - \frac{a}{r} \{ (1+r)^x - 1 \} .$$

(7). The same being supposed as in the last problem, only that the part of his capital vested in trade, yields r' per cent. per annum, but returns interest only once in n years. Required the amount of his funded stock after any number of years.

In this case the equation

$$P_{x+1} = R \,.\, P_x - a,$$

holds good for every value of x unless when $x + 1$ is a multiple of n (and therefore $x - n + 1$ such a multiple) when it changes to

$$P_{x+1} = R \,.\, P_x - (a - b),$$

where $b = n r' \,.\, \frac{A}{2}$. In this case then we have for our circulating equation,

$$o = P_{x+1} - R \,.\, P_x + \{ a \,.\, S_x + a \,.\, S_{x-1} + \ldots\ldots$$

$$\ldots\ldots (a - b) S_{x-n+1} \},$$

which integrated as in (9, Sect. 11.) gives

$$P_x = C \cdot R^x + \frac{a}{R-1} - \frac{b}{R^n - 1}$$

$$\{ S_x + R \cdot S_{x-1} + \ldots\ldots R^{n-1} \cdot S_{x-n+1} \} .$$

the constant C being determined as before, by taking $x = 0$, is found as follows :

$$C = \frac{A}{2} + \frac{b}{R^n - 1} - \frac{a}{R-1},$$

$$= \frac{A}{2}\left(1 + \frac{n r'}{R^n - 1}\right) - \frac{a}{R-1} .$$

(8). Suppose a merchant engaged in more than one such concern as those described in the two last problems. To determine his funded property after any time.

Let n represent the least time in which the interest of funded capital can be made readily to accrue, n', n'', &c. the intervals at which the several parts of his capital embarked in commerce return their interest, the least common measure of all their intervals n, n', n'', &c. or of such as differ from each other, being taken as the unit of time. Also let A', A'', &c. be the several parts of his capital so embarked, and r', r'', &c. the rates of interest they yield in the time 1. Lastly, let A be his original funded capital, P_x its amount after x such units of time have elapsed, r the rate of interest in the funds for the time 1, and a his uniform rate of expenditure in that time. Then we have

$$P_{x+1} = P_x - a,$$

unless when $x + 1$ is a multiple of either n, n', &c. in which several cases, the terms

$$n r \cdot P_x, \quad n' r' A' \text{ \&c.}$$

expressing the respective sums accruing as interest at these moments are to be added to the second member. Employing then the notation of (11, Sect. 11.) the circulating equation embracing all these cases is,

$$P_{x+1} = P_x(1 + nr \,.\, S_{x-n+1}^{(n)})$$
$$- a + n'r'A' \,.\, S_{x-n'+1}^{(n')} + \&c.$$

Take m = the product of n, n', &c. divided by all the greatest common measures of any two or more of them, and this equation is transformed to

$$0 = P_{x+1}$$
$$- P_x\{1 + nr(S_{x-n+1}^{(m)} + S_{x-2n+1}^{(m)} \ldots + S_{x-m+1}^{(m)})\} + a -$$
$$- n'r'A'\{S_{x-n'+1}^{(m)} + S_{x-2n'+1}^{(m)} \ldots + S_{x-m+1}^{(m)}\}$$
$$- n''r''A'', \&c.$$

Now, since

$$1 = S_x^{(m)} + S_{x-1}^{(m)} - \ldots S_{x-m+1}^{(m)},$$

and

$$a = a \,.\, S_x^{(m)} + \,.\, S_{x-1}^{(m)} + \&c.$$

If we take

$$a_1 = 1,\ a_2 = 1, \ldots a_n = 1 + nr,\ a_{n+1} = 1,\ a_{n+2} = 1, \ldots$$
$$\ldots\ldots a_{2n} = 1 - nr, \ldots \&c.$$

$$b_1 = 0,\ b_2 = 0, \ldots\ldots b_{n'} = n'r'A',\ b_{n'+1} = 0, \ldots\ldots$$
$$\ldots\ldots b_{2n'} = n'r'A',\ \&c.$$

$$c_1 = 0, \ldots\ldots c_{n''} = n''r''A'',\ c_{n''+1} = 0,\ \&c.\ \&c.$$

and finally

$$- a + b_1 + c_1 + \ldots = \alpha,\ - a + b_2 + c_2 + \ldots = \beta,\ \&c.\ \&c.$$

our equation will become

$$P_{x+1} - P_x\{a_1 S_x^{(m)} + \ldots a_m S_{x-m+1}^{(m)}\}$$
$$- \{\alpha \,.\, S_x^{(m)} \beta \,.\, S_{x-1}^{(m)} \ldots \kappa \,.\, S_{x-m+1}^{(m)}\} = 0,$$

whose integral is given at full length in (10, Sect. 11.)

In this case we have

$$N = (1 + nr)^{\frac{1}{n}},$$

and taking $P_0 = A$, we find for the arbitrary constant,

$$C = A - \frac{N}{1 - N}\left(\frac{\alpha}{a_1} + \frac{\beta}{a_1 a_2} + \dots \frac{\kappa}{a_1 \dots a_n}\right),$$

but till the particular values of n, n', &c. are assigned in numbers, no farther reductions in the form of this integral are practicable.

(9). A and B engage in play, on the following condition, viz. that whenever A wins a game, the stake shall be doubled for the next game, but whenever B wins, it shall be tripled. When they left off (after x games) it was found that they had won and lost alternately, A winning the first game. What are their respective gains and losses?

Put u_x to represent A's total gain at the end of the x^{th} game, and suppose P_x to equal the stake for which that game is played. Then, provided A win the x^{th} game, we have $P_{x+1} = 2\,P_x$, but if B win it $P'_{x+1} = 3\,P_x$. Now A wins it if x be odd, but B if even; hence in all cases, the numerical value of x being undetermined,

$$P_{x+1} = (3 \,.\, S_x + 2 \,.\, S_{x-1})\,P_x; \qquad (a.)$$

where $S_x = \frac{1}{2}$ the sum of the x^{th} powers of the roots of $z^2 - 1 = 0$.

Again, $\Delta\, u_x$ is A's gain or loss by the event of the $(x+1)^{th}$ game. It is evidently equal in value to the stake for which that game was played (P_{x+1}) being a *gain* if A win, or $x+1$ be odd, but a *loss* if he lose, or $x+1$ be even, hence

$$u_{x+1} - u_x = (S_x - S_{x-1})\,P_{x+1}; \qquad (b.)$$

It only remains therefore to integrate the equations (a), (b). Now the equation (a) is that of (8, Sect. 12.), whence, determining the constant by taking P_1 the first stake for a given quantity, we find

$$P_x = P_1 . \sqrt{2} . \{ \sqrt{2} . S_x + \sqrt{3} . S_{x-1} \} (\sqrt{6})^{x-2}.$$

This given, we get, by substituting the value of P_{x+1} in (b),

$$u_{x+1} - u_x = P_1 . \sqrt{2} \{ \sqrt{3} . S_x - \sqrt{2} . S_{x-1} \} . (\sqrt{6})^{x-1}.$$

To integrate this, we take $u_x = A_x . S_x + B_x . S_{x-1}$, which gives by substitution and equation of like terms

$$B_{x+1} - A_x = P_1 . (\sqrt{6})^x$$

$$A_{x+1} - B_x = - 2 P_1 . (\sqrt{6})^{x-1}, \text{ or}$$

$$A_{x+2} - B_{x+1} = - 2 P_1 . (\sqrt{6})^x.$$

Adding this and the first together, we find

$$A_{x+2} - A_x = - P_1 . (\sqrt{6})^x,$$

and integrating

$$A_x = C S_x + C' S_{x-1} - \frac{P_1}{5} . (\sqrt{6})^x,$$

consequently

$$B_x = C' S_x + C S_{x-1} + \frac{4}{5} . P_1 (\sqrt{6})^{x-1},$$

and

$$u_x = C - \frac{1}{5} P_1 . \{ S_x . (\sqrt{6})^x - 4 S_{x-1} . (\sqrt{6})^{x-1} \}.$$

But $u_0 = 0$, hence $C = \frac{1}{5} P$, and finally

$$u_x = \tfrac{1}{5} P_1 \{ 1 - (\sqrt{6})^x . S_x + 4 . (\sqrt{6})^{x-1} . S_{x-1} \},$$

for the total amount of A's gain or B's loss at the end of their play.

(10). The last problem may be generalized by supposing that when A wins, the stake of the succeeding game shall become any function whatever of the former stake, and of the number of games elapsed since the beginning, and when B wins, any other functions. A and B winning alternately, what is the total amount of A's gain or loss?

Here we have $P_{x+1} = f(P_x, x)$,

when x is odd, and $= f_1(P_x, x)$ when even, so that in general

$$P_{x+1} = S_x . f_1(P_x, x) + S_{x-1} . f(P_x, x).$$

To integrate this, or at least to clear it of its circulating form, we take

$$P_x = A_x . S_x + B_x . S_{x-1}.$$

Then since $f(P_x, x)$ is a function of S_x, S_{x-1} it is reducible to the general form of all circulating functions (by 12, Sect. 12.) and in fact it becomes

$$f(P_x, x) = f(A_x, x) . S_x + f(B_x, x) S_{x-1},$$

similarly,

$$f_1(P_x, x) = f_1(A_x, x) . S_x + f_1(B_x, x) . S_{x-1},$$

therefore

$$S_x \{ B_{x+1} - f_1(A_x, x) \} + S_{x-1} A_{x+1} - f(B_x, x) \} = 0,$$

which resolves itself into the two

$$B_{x+1} = f_1(A_x, x) \qquad A_{x+1} = f(B_x, x),$$

That is,

$$A_{x+2} = f \{ f_1(A_x, x), x \}$$

an equation of the second order whose integration suffices to give A_x. If we put $x = 2z$ and $A_{2z} = A_z'$, this reduces itself to the first order, after which it depends on the particular forms of f, f_1, whether the integration be practicable or not. A very extensive case of integrability is when

$$f(P_x, x) = a_x \,.\, P_x + b_x$$

$$f_1(P_x, x) = \alpha_x \,.\, P_x + \beta_x.$$

In this case the equation for finding A_z' is linear of the first order, viz.

$$A'_{z+1} = a_x \alpha_x \,.\, A_z' + (b_x + a_x \beta_x)$$

$$= a_{2z} \alpha_{2z} \,.\, A_z' + (b_{2z} + a_{2z} \beta_{2z}.$$

In like manner, we may proceed when instead of winning and losing alternately, the players win and lose in any other regular order.

(11). Let there be a series of quantities A, B, C, &c. derived from one another by the following law,

$$A = a, \; B = \frac{1 - A}{2}, \; C = \frac{1 - 2B}{3}, \; D = \frac{1 - 3C}{4}, \; \&c.$$

Required the general term of the series.

Call the x^{th} term u_x, then the $(x + 1)^{\text{th}}$ being u_{x+1} we have

$$u_{x+1} = \frac{1 - x u_x}{x + 1},$$

$$(x + 1) \,.\, u_{x+1} + x u_x = 1$$

an equation of differences which integrated give,

$$x u_x = \tfrac{1}{2} + C \,.\, (-1)^x$$

and determining C so that $u_1 = a$

$$u_x = \frac{1}{2x} + \frac{(2a-1).(-1)^{x+1}}{2x}.$$

(12). Suppose we have

$$A = a, \; B = 2A^2 - 1, \; C = 2B^2 - 1, \; \&c.$$

Required the x^{th} term of this series.

Here $$u_{x+1} = 2u_x^2 - 1,$$

an equation which integrated gives

$$u_x = \tfrac{1}{2}\{(a + \sqrt{a^2-1})^{2^{x-1}} + (a - \sqrt{a^2-1})^{2^{x-1}}\},$$

the constant being properly determined. In the same way we might proceed, if we had

$$A = a, \; B = f(A), \; C = f(B), \; D = f(C) \; \&c.;$$

(13). To integrate the differential expression

$$\int \frac{x^{2n}\,dx}{\sqrt{(1-x^2)}}.$$

Assume it equal to F_n (being a function of n to be determined). Then, integrating by parts,

$$F_n = \int x^{2n-1} . \frac{x\,dx}{\sqrt{(1-x)^2}},$$

$$= -x^{2n-1}\sqrt{(1-x^2)} + (2n-1)\int x^{2n-2}\,dx . \sqrt{(1-x^2)},$$

or, putting $P_n = x^{2n-1}\sqrt{(1-x^2)}$, and writing $\frac{1-x^2}{\sqrt{(1-x)^2}}$ for $\sqrt{(1-x^2)}$

$$F_n = -P_n + (2n-1)(F_{n-1} - F_n).$$

That is,

$$F_n - \frac{2n-1}{2n} . F_{n-1} = -\frac{P_n}{2n}; \qquad (a).$$

an equation of differences (n being the independent variable) whose integral is

$$F_n = \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{2 \cdot 4 \ldots\ldots (2n)} \times$$

$$\left\{ C - \Sigma \frac{2 \cdot 4 \ldots\ldots (2n)}{1 \cdot 3 \cdot 5 \ldots\ldots (2n+1)} P_{n+1} \right\}.$$

The integration denoted by the sign Σ not being practicable we must (as in 3, Sect. 4.) write at full length the series of which the integral consists, viz.

$$\frac{1}{1} P_1 + \frac{2}{1 \cdot 3} P_2 + \ldots \frac{2 \cdot 4 \ldots (2n-2)}{1 \cdot 3 \ldots (2n-1)} P_n,$$

and determining C by the condition

$$F_0 = \int^x \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x,$$

we get, restoring the values of P_1, P_2, &c.

$$F_n = \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots (2n)} \left\{ \sin^{-1} x - \left(\frac{x}{1} + \frac{2x^3}{1 \cdot 3} + \right.\right.$$

$$\left.\left. \ldots\ldots \frac{2 \cdot 4 \ldots (2n-2)}{1 \cdot 3 \ldots (2n-1)} x^{2n-1} \right) \cdot \sqrt{(1-x^2)} \right\}.$$

If we only require the value of the integral between the limits $x = 0$, and $x = 1$, since $x^{2n-1} \cdot \sqrt{(1-x^2)}$ vanishes at both the limits, we have $P_n = 0$, and our equation of differences (a) is simply

$$F_n - \frac{2n-1}{2n} F_{n-1} = 0,$$

which gives

$$F_n = C \times \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots (2n)},$$

but $C = \int^{\circ} \frac{dx}{\sqrt{(1 - x^2)}} = \frac{\pi}{2}$, between the same limits; so that

$$F_n = \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots (2n)} \cdot \left(\frac{\pi}{2}\right);$$

and in the same way may all similar instances in which integrals are reduced by successive steps to a less and less degree of complication till at last they are brought to a known form, be treated without going through the process of continuation, by reducing them to equations of differences.

EXAMPLES

OF THE

Solutions

OF

FUNCTIONAL EQUATIONS.

BY

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NOTICE.

THE object of the following Examples of Functional Equations, is to render a subject of considerable interest, more accessible to mathematical students, than it has hitherto been. It is, perhaps, that subject of all others, which most requires the assistance of particular instances, in order fully to comprehend the meaning of its symbols, which are of the most extreme generality; that assistance is also more particularly required in this branch of science, in consequence of its never yet having found its way into an Elementary Treatise.

Oct. 20, 1820.

OF

FUNCTIONAL EQUATIONS.

If a function α is of such a form, that, when it is twice performed on a quantity, the result is the quantity itself, or if $\alpha^2(x) = x$, then it is called a periodic function of the second order, if $\alpha^n(x) = x$, then it is termed a periodic function of the n^{th} order, thus when $\alpha(x) = a - x$ the second function, or

$$\alpha(\alpha x) = \alpha(a - x) = a - (a - x) = a - a + x = x.$$

If $\alpha(x) = \dfrac{1}{1 - x}$,

$$\text{then } \alpha^2 x = \alpha(\alpha x) = \frac{1}{1 - \dfrac{1}{1 - x}} = \frac{1 - x}{1 - x - 1} = \frac{x - 1}{x},$$

and

$$\alpha^3 x = \alpha^2 \alpha x = \frac{\alpha x - 1}{\alpha x} = \frac{\dfrac{1}{1 - x} - 1}{\dfrac{1}{1 - x}} = 1 - \overline{1 - x} = x,$$

the first of these examples is a periodic function of the second, the last is a periodic function of the third order.

Prob. 1. To find periodic functions of the second order.

Since such functions must satisfy the equation $\psi^2 x = x$, we have

$$\psi x = \psi^{-1} x,$$

or ψ must be such a function, that it shall be the same as its inverse; if therefore $y = \psi x$, we have also $x = \psi^{-1} y = \psi y$, or if x and y are connected by some equation, it must be symmetrical relative to x and y; y or ψx must then be determined from the equation

$$* F\{\overline{x}, \overline{\psi x}\} = 0,$$

for instance, if $x + \psi x - a = 0$, $\psi x = a - x$,

or if $x \psi x = a^2$, $\psi x = \frac{a^2}{x}$.

Another method of determining such functions is as follows: since ψx is of such a form that $\psi^2 x = x$ any symmetrical function of x and ψx remain constant when x is changed into ψx thus

$$F\{\overline{x}, \overline{\psi x}\} \text{ becomes } F\{\overline{\psi x}, \overline{\psi^2 x}\} = F\{\overline{\psi x}, \overline{x}\},$$

if therefore, we can find any particular solution of the equation $\psi^2 x = x$, containing an arbitrary constant we may substitute such a function for it, but $\psi x = a - x$ is a particular solution therefore

$$\psi x = F(\overline{x}, \overline{\psi x}) - x,$$

or

$$x + \psi x = F(\overline{x}, \overline{\psi x}),$$

and by changing the arbitrary function into another of the same form, we find

$$F\{\overline{x}, \overline{\psi x}\} = 0,$$

as before.

These two methods of determining periodic functions of the second order, are not so convenient as a third process which can be extended to all orders.

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* Bars placed above quantities under the functional sign, indicate that the function is symmetrical relative to those quantities.

Assume $\psi\, x = \phi^{-1} f \phi\, x$, then

$$\psi^2 x = \phi^{-1} f \phi \phi^{-1} f \phi\, x = \phi^{-1} f^2 \phi\, x,$$

this must be equal to x or

$$\phi^{-1} f^2 \phi\, x = x,$$

this equation will be fulfilled if $f^2 v = v$, or if f is a particular solution, and if also ϕ^{-1} is such an inverse function that $\phi^{-1} \phi\, v = v$. If therefore ϕ is arbitrary, and f is a particular solution of $f^2 x = x$, then the solution of $\psi^2 x = x$ is

$$\psi\, x = \phi^{-1} f \phi\, x.$$

Ex. Let $f x = \frac{a}{x}$, then $\psi\, x = \phi^{-1}\left(\frac{a}{\phi\, x}\right)$,

if $f(x) = \frac{a - b x}{b + c x}$, $\quad \psi\, x = \phi^{-1}\left(\frac{a - b \phi\, x}{b + c \phi\, x}\right)$;

from these may easily be derived the following periodic functions of the second order,

$$\psi\, x = a - x \qquad \psi\, x = \frac{x}{x - 1}$$

$$\psi\, x = \frac{x - 2}{x - 1} \qquad \psi\, x = \frac{a^2}{x}$$

$$\psi\, x = \frac{1 - x}{1 + x} \qquad \psi\, x = \sqrt{1 - x^2}$$

$$\psi\, x = \frac{x + 1}{x - 1} \qquad \psi\, x = \frac{x}{\sqrt{x^2 - 1}},$$

$$\psi\, x = \tan^{-1}\left(\frac{\sin (a - x)}{\cos a . \cos x}\right) \qquad \psi\, x = \log (a - \epsilon^x)$$

$$\psi\, x = (a^n - x^n)^{\frac{1}{n}} \qquad \psi\, x = x - \log (\epsilon^x - 1)$$

$$\psi\, x = \frac{x}{(x^n - a^n)^{\frac{1}{n}}} \qquad \psi\, x = \tan^{-1}(a - \tan x)$$

or if x and y are connected by some equation, it must be symmetrical relative to x and y; y or ψx must then be determined from the equation

$$* F\{\overline{x}, \overline{\psi x}\} = 0,$$

for instance, if $x + \psi x - a = 0$, $\psi x = a - x$,

or if $x \psi x = a^2$, $\psi x = \frac{a^2}{x}$.

Another method of determining such functions is as follows: since ψx is of such a form that $\psi^2 x = x$ any symmetrical function of x and ψx remain constant when x is changed into ψx thus

$$F\{\overline{x}, \overline{\psi x}\} \text{ becomes } F\{\overline{\psi x}, \overline{\psi^2 x}\} = F\{\overline{\psi x}, \overline{x}\},$$

if therefore, we can find any particular solution of the equation $\psi^2 x = x$, containing an arbitrary constant we may substitute such a function for it, but $\psi x = a - x$ is a particular solution therefore

$$\psi x = F(\overline{x}, \overline{\psi x}) - x,$$

or

$$x + \psi x = F(\overline{x}, \overline{\psi x}),$$

and by changing the arbitrary function into another of the same form, we find

$$F\{\overline{x}, \overline{\psi x}\} = 0,$$

as before.

These two methods of determining periodic functions of the second order, are not so convenient as a third process which can be extended to all orders.

---

* Bars placed above quantities under the functional sign, indicate that the function is symmetrical relative to those quantities.

Assume $\psi x = \phi^{-1} f \phi x$, then

$$\psi^2 x = \phi^{-1} f \phi \phi^{-1} f \phi x = \phi^{-1} f^2 \phi x,$$

this must be equal to x or

$$\phi^{-1} f^2 \phi x = x,$$

this equation will be fulfilled if $f^2 v = v$, or if f is a particular solution, and if also ϕ^{-1} is such an inverse function that $\phi^{-1} \phi v = v$. If therefore ϕ is arbitrary, and f is a particular solution of $f^2 x = x$, then the solution of $\psi^2 x = x$ is

$$\psi x = \phi^{-1} f \phi x.$$

Ex. Let $f x = \frac{a}{x}$, then $\psi x = \phi^{-1} \left(\frac{a}{\phi x}\right)$,

if $f(x) = \frac{a - b x}{b + c x}$, $\psi x = \phi^{-1} \left(\frac{a - b \phi x}{b + c \phi x}\right)$;

from these may easily be derived the following periodic functions of the second order,

$$\psi x = a - x \qquad \psi x = \frac{x}{x - 1}$$

$$\psi x = \frac{x - 2}{x - 1} \qquad \psi x = \frac{a^2}{x}$$

$$\psi x = \frac{1 - x}{1 + x} \qquad \psi x = \sqrt{1 - x^2}$$

$$\psi x = \frac{x + 1}{x - 1} \qquad \psi x = \frac{x}{\sqrt{x^2 - 1}},$$

$$\psi x = \tan^{-1} \left(\frac{\sin (a - x)}{\cos a . \cos x}\right) \qquad \psi x = \log (a - \epsilon^x)$$

$$\psi x = (a^n - x^n)^{\frac{1}{n}} \qquad \psi x = x - \log (\epsilon^x - 1)$$

$$\psi x = \frac{x}{(x^n - a^n)^{\frac{1}{n}}} \qquad \psi x = \tan^{-1} (a - \tan x)$$

Prob. 2. Required periodic functions of the third order, or such as fulfil the equation $\psi^3 x = x$.

Assume $\psi x = \phi^{-1} f \phi x$, then the equation becomes

$$\psi^3 x = \phi^{-1} f \phi \phi^{-1} f \phi \phi^{-1} f \phi x = \phi^{-1} f^3 \phi x = x,$$

which will be verified if $f(v)$ is a particular solution of $f^3 v = v$, and if ϕ^{-1} is such an inverse value that $\phi^{-1} \phi v = v$: hence the solution of the equation is

$$\psi x = \phi^{-1} f \phi x,$$

one solution is $\frac{1}{1 - x}$ and hence $\psi x = \phi^{-1} \left(\frac{1}{1 - \phi x} \right)$

more particular cases are

$$\psi x = \frac{a^2}{a - x} \qquad \psi x = \frac{1 + x}{1 - 3x}$$

$$\psi x = \frac{a^2}{ac - c^2 x} \qquad \psi x = \frac{\sqrt{a x^2 - a^2}}{x}$$

$$\psi x = \frac{a x - a^2}{x} \qquad \psi x = \frac{1}{1 - x}$$

$$\psi x = \left(\frac{a^2}{a - x^n} \right)^{\frac{1}{n}} \qquad \psi x = -\log (1 - \varepsilon^x)$$

$$\psi x = \frac{(a x^n - a^2)^{\frac{1}{n}}}{x} \qquad \psi x = \log (a \varepsilon^x - a^2) - x$$

$$\psi x = \frac{a + b x}{c - \frac{a^2 + b c + c^2 x}{a}} \qquad \psi x = \log (\varepsilon^x - \varepsilon^c) - x + c.$$

Prob. 3. To find periodic functions of the n^{th} order, or to solve the equation $\psi^n x = x$.

Assume as before $\psi x = \phi^{-1} f \phi x$ then it becomes

$$\phi^{-1} f \phi \phi^{-1} f \phi \ldots \phi^{-1} f \phi x = \phi^{-1} f^n \phi x = x,$$

which is verified if f is a particular solution of $f^n x = x$, and if ϕ^{-1} is such an inverse function that $\phi^{-1}\phi x = x$.

It now remains to find particular solutions of $\psi^n x = x$ which may be accomplished in the following manner: let fx represent $\frac{a+bx}{c+dx}$ then the n^{th} function will be of the same form, or

$$f^n(x) = \frac{A_n + B_n x}{C_n + D_n x},$$

where A_n, B_n, C_n, D_n, are functions of a, b, c, d, and n, these may be so determined that $D_n = 0$, $A_n = 0$ and $B_n = C_n$ all which conditions are satisfied, if

$$d = -\frac{b^2 - 2bc\cos\frac{2k\pi}{n} + c^2}{\left(2 + 2\cos\frac{2k\pi}{n}\right)a}$$

hence

$$\phi x = \phi^{-1}\left\{\frac{a + b\phi x}{c - \frac{b^2 - 2bc\cos\frac{2k\pi}{n} + c^2}{\left(2 + 2\cos\frac{2k\pi}{n}\right)a}\phi x}\right\}$$

a more detailed account of this method of solution may be found in a paper by Mr. Horner in the Annals of Philosophy, Nov. 1817.

Instances of $\psi^4 x = x$ are

$$\psi x = \frac{1}{2}\frac{1}{1-x} \qquad \psi x = \frac{1+x}{1-x}$$

$$\psi x = \frac{2}{2-x} \qquad \psi x = \frac{2a^2}{2ac - c^2 x}$$

$$\psi x = 2\frac{x-1}{x} \qquad \psi x = \frac{a + bx}{c - \frac{b^2 + c^2}{2a}x}$$

$$\psi x = \frac{\sqrt{2}}{\sqrt{2 - x^2}} \qquad \psi x = -\sqrt{\frac{1 - x^2}{1 + x^2}}$$

$$\psi x = \frac{(2 x^n - 2)^{\frac{1}{n}}}{x}$$

$$\psi x = \log 2 - x + \log (\epsilon^x - 1).$$

All those cases which satisfy the equation $\psi^2 x = x$, also fulfil that of $\psi^4 x = x$, as well as all those which fulfil any of these equations $\psi^2 x = - x$, $\psi^2 x = \frac{1}{x}$, or more generally $\psi^2 x = \alpha x$, where αx is a particular solution of the equation $\psi^2 x = x$.

The following particular cases satisfy the equation $\psi^6 x = x$.

$$\psi x = \frac{1}{3(1 - x)} \qquad \psi x = \frac{3 x - 1}{3 x}$$

$$\psi x = \frac{3}{3 - x} \qquad \psi x = \frac{3 a^2}{3 a c - c^2 x}$$

$$\psi x = 3 \frac{x - 1}{x} \qquad \psi x = \frac{3 + 3 x}{3 - x}$$

$$\psi x = \frac{a + b x}{c - \frac{b^2 - b c + c^2}{3 a} x}$$

$$\psi x = \frac{1}{x} \left(x^n - \frac{1}{3} \right)^{\frac{1}{n}}$$

$$\psi x = \log 3 - x + \log (\epsilon^x - 1).$$

The principle on which the solution of the functional equation $F \{ x, \psi x, \psi \alpha x \} = 0$ depends, where $\alpha^2 x = x$, is that by substituting αx for x we have another equation $F \{ \alpha x, \psi \alpha x, \psi x \} = 0$, between which and the given equation we may eliminate $\psi \alpha x$ and the result will be the value of ψx a few examples will illustrate this method.

(1). Given $\psi(x) + a\,\psi(-x) = x^n$

by putting $-x$ for x this becomes

$$\psi(-x) + a\,\psi(x) = (-x)^n,$$

and eliminating $\psi(-x)$, we have

$$\psi x - a^2\,\psi x = x^n - a(-x)^n,$$

hence

$$\psi x = \frac{1-(-1)^n a}{1-a^2}\,x^n.$$

(2). Given $\psi x - a\,\psi \frac{1}{x} = \epsilon^x$

put $\frac{1}{x}$ for x, $\psi \frac{1}{x} - a\,\psi x = \epsilon^{\frac{1}{x}}$

and

$$\psi x - a\,\epsilon^{\frac{1}{x}} - a^2\,\psi x = \epsilon^x,$$

$$\psi x = \frac{\epsilon^x + a\,\epsilon^{\frac{1}{x}}}{1-a^2}$$

(3). Given $(\psi x)^2 . \psi \frac{1-x}{1+x} = c^2 x$

put $\frac{1-x}{1+x}$ for x, it becomes

$$\left(\psi \frac{1-x}{1+x}\right)^2 . \psi x = c^2\,\frac{1-x}{1+x},$$

eliminating $\psi \frac{1-x}{1+x}$ by means of the former, we find

$$\psi x = \left(\frac{1+x}{1-x}\,c^2\,x^2\right)^{\frac{1}{3}}.$$

(4). Given $\psi x + \frac{1}{1-x^2}\psi\sqrt{(1-x^2)} = 1 + x^2$

putting $\sqrt{(1-x^2)}$ for x, we have

$$\psi\sqrt{(1-x^2)} + \frac{1}{x^2}\psi x = 2 - x^2,$$

and substituting this value of $\psi\sqrt{1-x^2}$ in the former equation

$$\psi x + \frac{2-x^2}{1-x^2} - \frac{1}{x^2-x^4}\psi x = 1 + x^2,$$

hence

$$\left(\frac{x^2-x^4-1}{x^2-x^4}\right)\psi x = 1 + x^2 - \frac{2-x^2}{1-x^2} = \frac{-1+x^2-x^4}{1-x^2}$$

and $\psi x = x^2$.

(5). Given $\frac{\psi x}{1+\psi x} + x\frac{\psi(-x)}{1+\psi(-x)} = 1$

put $\psi_1 x = \frac{\psi x}{1+\psi x}$ thus the equation becomes

$\psi_1 x + x\psi_1(-x) = 1$, and changing x into $-x$ we have $\psi_1(-x) - x\psi_1(x) = 1$, by which eliminating $\psi(-x)$ from the former, we find

$$\psi_1 x = \frac{1-x}{1+x^2},$$

hence

$$\psi x = \frac{\psi_1 x}{1-\psi_1 x} = \frac{1-x}{x+x^2}.$$

(6). Given $\psi x + \frac{1+x}{x}\psi\frac{1}{x} = c,$

putting $\frac{1}{x}$ for x this becomes

$$\psi \frac{1}{x} + (1 + x) \psi x = x$$

and by eliminating $\psi \frac{1}{x}$, we have

$$\psi x = \frac{1}{1 + x + x^2} c$$

(7). Given $\psi x + x \psi (1 - x) = 1$,
putting $1 - x$ for x, we have

$$\psi (1 - x) + (1 - x) \psi (x) = 1,$$

whence, by elimination,

$$\psi x = \frac{1 - x}{1 - x(1 - x)} = \frac{1 - x}{1 - x + x^2}.$$

(8). Given $\frac{\psi x}{\psi x - x} + x \frac{\psi (1 - x)}{\psi (1 - x) + x - 1} = 1$,

put $\psi_1 x = \frac{\psi x}{\psi x - x}$, then will $\psi_1 (1 - x) = \frac{\psi (1 - x)}{\psi (1 - x) + x - 1}$,
and the equation becomes

$$\psi_1 x + x \psi_1 (1 - x) = 1,$$

the same as in the last example; let $f x$ represent the solution there found, then

$$\psi_1 x = f x = \frac{\psi x}{\psi x - x},$$

whence

$$\psi x = \frac{x f x}{f x - 1},$$

if we take for $f x$ its value $\frac{1 - x}{1 - x + x^2}$, we have

$$\psi x = \frac{x - 1}{x}.$$

In case the equation is symmetrical with regard to ψx and $\psi a x$, the process of elimination apparently becomes illusory. By a peculiar artifice this difficulty may be overcome, and it happens rather singularly that in all these cases, the solution which is so obtained contains an arbitrary function, and in general the solution is the most extensive which the question admits of.

(9). Given $\psi x = \psi \frac{1}{x}$.

If we put $\frac{1}{x}$ for x, this is changed into $\psi \frac{1}{x} = \psi x$, the same as the given equation; it is therefore impossible to eliminate.

Let us now suppose $\psi x = a \psi \frac{1}{x} + b$,

which becomes the given equation when $a = 1$ and $b = 0$. By putting $\frac{1}{x}$ for x this is changed into

$$\psi \frac{1}{x} = a \psi x + b,$$

and eliminating $\psi \frac{1}{x}$, we have

$$\psi x = \frac{a b + b}{1 - a^2} = \frac{b}{1 - a},$$

if $b = 0$ and $a = 1$, this becomes a vanishing fraction whose value is any constant quantity c, and we have $\psi x = c$, which fulfils the equation. This is a very limited solution, but the following plan will lead us to much more general ones.

Take the equation

$$\psi x = a \psi \frac{1}{x} + v \phi x,$$

which coincides with the given one when $v = 0$ and $a = 1$; also ϕx is any arbitrary function of x; putting $\frac{1}{x}$ for x, we have

$$\psi \frac{1}{x} = a \psi x + v \phi \frac{1}{x},$$

and by elimination,

$$\psi x = \frac{a \phi \frac{1}{x} + \phi x}{1 - a^2} v.$$

Let a become $1 + 0$ and v become 0 at the same time, then

$$\frac{0}{1 - (1 + 2 . 0 + 0^2)} = \frac{0}{-2 . 0 + 0^2} = \frac{1}{-2 + 0} = -\frac{1}{2}$$

and the solution becomes

$$\psi x = -\frac{\phi \frac{1}{x} + \phi x}{2},$$

or changing the arbitrary function

$$\psi x = \phi x + \phi \frac{1}{x},$$

in which ϕ is indefinite.

This solution is, in fact, nothing more than an arbitrary symmetrical function of x and $\frac{1}{x}$, and may be expressed thus

$$\psi x = \chi \left(\overline{x}, \overline{\frac{1}{x}} \right).$$

Precisely the same course of reasoning will produce the solutions of the following equation.

(10). $\psi (x) = \psi (a - x)$

$$\psi x = \chi (\overline{x}, \overline{a - x}).$$

(11). $\psi (x) = \psi \left(\frac{1 - x}{1 + x} \right)$, $\psi x = \chi \left\{ \overline{x}, \overline{\frac{1 - x}{1 + x}} \right\}$.

(12). $\psi(x) = \psi\left(\frac{x}{\sqrt{(x^2-1)}}\right)$, $\psi x = \chi\left\{\bar{x}, \overline{\frac{x}{\sqrt{(x^2-1)}}}\right\}$.

(13). $\psi\left(\frac{x}{2-x}\right) = \psi(1-x)$, $\psi x = \chi\left\{\bar{x}, \overline{\frac{1-x}{1+x}}\right\}$.

(14). $\psi x = \psi(a x)$, $\psi x = \chi(x, a x)$, where $a^2 x = x$.

(15). The objection which has just been stated occurs in the equation $\psi x + \psi\left(\frac{x}{x-1}\right) = c$,

and a similar mode of proceeding will obviate it. The given equation is a particular case of

$$\psi x + a\,\psi\left(\frac{x}{x-1}\right) = c + v\,\phi\, x,$$

with which it coincides, if $a=1$ and $v=0$; putting $\frac{x}{x-1}$ for x in this, we have

$$\psi\left(\frac{x}{x-1}\right) + a\,\psi\, x = c + v\,\phi\left(\frac{x}{x-1}\right),$$

and elimination produces

$$\psi x = \frac{c - a c}{1 - a^2} + \left\{\phi x - a\,\phi\left(\frac{x}{x-1}\right)\right\}\frac{v}{1-a^2}$$

If $v=0$ and $a=1$, this gives

$$\psi x = \frac{c}{2} + \phi x - \phi\left(\frac{x}{x-1}\right)$$

in which the function ϕ has been changed into another similar one.

16. Given $\psi(1+x) + \psi(1-x) = 1 - x^2$,

put $x-1$ for x, then

$$\psi x + \psi(2 - x) = 1 - (x - 1)^2 = 2x - x^2,$$

this is a particular case of the equation

$$\psi x + a\psi(2 - x) = 2x - x^2 + v\phi x,$$

with which it agrees if $v=0$ and $a=1$; changing x into $2-x$ and eliminating $\psi(2 - x)$ from the result, we find

$$\psi x = \frac{2x - x^2 - a(2x - x^2)}{1 - a^2} + \{\phi x - a\phi(2 - x)\}\frac{v}{1 - a^2}.$$

or

$$\psi x = \frac{2x - x^2}{1 + a} + \{\phi x - a\phi(2 - x)\}\frac{v}{1 - a^2},$$

If $a=1$ and $v=0$, we have

$$\psi x = \frac{2x - x^2}{2} + \phi x - \phi(2 - x).$$

(17.) Given $\dfrac{1}{x + \psi x} + \dfrac{x}{1 + x\psi\frac{1}{x}} = 2,$

put $\dfrac{1}{x + \psi x} = \psi_1 x$ then $\dfrac{1}{\frac{1}{x} + \psi\frac{1}{x}} = \psi_1\dfrac{1}{x} = \dfrac{x}{1 + x\psi\frac{1}{x}},$

and the equation becomes

$$\psi_1 x + \psi_1\frac{1}{x} = 2,$$

whose solution may be found by the method just explained to be

$$\psi_1 x = 1 + \phi x - \phi\frac{1}{x},$$

hence

$$\psi x = \frac{1}{1 + \phi x - \phi\frac{1}{x}} - x.$$

(18). Required the equation of that class of curves which possess the following property, (Part IV. Fig. 1.) a given abscissa $AB = a$ being taken, then the product of any two ordinates at equal distances from B, shall always be equal to the square of the abscissa a. If $y = \psi x$ represent the equation of the curve, then the condition expressed analytically is

$$\psi(a - x) \,.\, \psi(a + x) = a^2.$$

Putting $a - x$ for x, and then $\log \psi(x) = \psi_1 x$, we have

$$\psi_1 x + \psi_1 (2a - x) = 2 \log a,$$

whose solution is

$$\psi_1 x = \log a + \phi x - \phi(2a - x),$$

hence

$$\log \psi x = \log a + \phi x - \phi(2a - x),$$

and

$$\psi x = \varepsilon^{\log a} \times \varepsilon^{\phi x} \times \varepsilon^{-\phi(2a - x)} = \frac{a\, \varepsilon^{\phi x}}{\varepsilon^{\phi(2a - x)}}$$

and changing the arbitrary function ϕ into $\log \phi$,

$$\psi x = a \frac{\phi x}{\phi(2a - x)},$$

and the class of curves are comprehended in the equation

$$y = \frac{a \phi x}{\phi(2a - x)}.$$

(19). Given the equation

$$\{\psi x\}^2 + \left\{\psi\left(\frac{\pi}{2} - x\right)\right\}^2 = 1.$$

This is the equation on which the composition of forces is made to depend in the Mecanique Cœleste, p. 5.

Put $\psi_1 x$ for $(\psi x)^2$ then it becomes

$$\psi_1 x + \psi_1\left(\frac{\pi}{2} - x\right) = 1.$$

which is a particular case of

$$\psi_1 x + a\,\psi_1\left(\frac{\pi}{2} - x\right) = 1 + v\,\phi\,x.$$

Substituting $\frac{\pi}{2} - x$ for x, this gives

$$\psi_1\left(\frac{\pi}{2} - x\right) + \psi_1 x = 1 + v\,\phi\left(\frac{\pi}{2} - x\right),$$

and eliminating $\psi_1\left(\frac{\pi}{2} - x\right)$, we have

$$\psi\,x = \frac{1}{1+a} - \frac{\phi\,x - a\,\phi\left(\frac{\pi}{2} - x\right)v}{1 - a^2}$$

making $a = 1$ and $v = 0$, and changing $\phi\,x$ in $2\,\phi\,x$, we have

$$\psi_1 x = \frac{1}{2} - \phi\,x + \phi\left(\frac{\pi}{2} - x\right)$$

and therefore

$$\psi\,x = \sqrt{\frac{1}{2} - \phi\,x + \phi\left(\frac{\pi}{2} - x\right)}$$

In case the coefficient of $\psi\,a\,x$ in the equation $\psi\,x + f\,x\,\psi\,a\,x = f\,x$, is of such a form that $f\,x\,.f\,a\,x = 1$, the denominator will vanish, and we must then have recourse to an artifice similar to that which has already been explained.

(20). Ex. 1. Let $\psi\,x + x^{2n}\,\psi\,\frac{1}{x} = x^n$.

put $\psi\,x + (x^{2n} + v\,\phi\,x)\,\psi\,\frac{1}{x} = x^n$

which coincides with the given equation if $v=0$; then changing x into $\frac{1}{x}$, we have

$$\psi\frac{1}{x} + \left(x^{-2n} + v\,\phi\frac{1}{x}\right)\psi x = x^{-n}$$

and by elimination

$$\psi x = \frac{x^n - (x^{2n} + v\,\phi\, x)\,x^{-n}}{1-(x^{2n}+v\,\phi\, x)\left(x^{-2n} + v\,\phi\frac{1}{x}\right)}$$

$$= \frac{x^{-n}\,\phi\, x}{x^{2n}\,\phi\frac{1}{x} + x^{-2n}\,\phi\, x + v\,\phi\, x\,.\,\phi\frac{1}{x}}$$

and when v vanishes,

$$\psi x = \frac{x^{-n}\,\phi\, x}{x^{2n}\,\phi\frac{1}{x} + x^{-2n}\,\phi\, x}, \qquad (a).$$

The equation in this example may be solved differently, as follows. Multiply by x^{-n} and it becomes

$$x^{-n}\,\psi x + x^n\,\psi\frac{1}{x} = 1,$$

put $\psi_1 x = x^{-n}\,\psi x$, then

$$\psi_1 x + \psi_1\frac{1}{x} = 1,$$

whose general solution found by a process already explained is

$$\psi_1 x = \frac{1}{2} - \phi\, x + \phi\frac{1}{x},$$

hence

$$\psi x = \frac{x^n}{2} - x^n\,\phi\, x + x^n\,\phi\frac{1}{x}.$$

This solution differs in form from that which was previously found, but it may be proved to be the same by the following substitution; since ϕ is quite an arbitrary function

$$\phi x = x^{2n}\left(1 - 2\,\phi_1 x + 2\,\phi_1 \frac{1}{x}\right),$$

this gives $x^{2n}\,\phi\frac{1}{x} + x^{-2n}\,\phi\, x = 2$ and (a) becomes

$$\psi x = \frac{x^n}{2} - x^n \phi_1 x + x^n \phi_1 \frac{1}{x},$$

exactly as the last solution.

(21). Given

$$\left(\frac{\psi x + 1}{\psi x - 1}\right)^n - \left(\frac{\psi\left(\frac{x}{x-1}\right) + 1}{\psi\left(\frac{x}{x-1}\right) - 1}\right)^n = \frac{(x-1)^2 - 1}{x - 1};$$

put $\psi_1 x = \dfrac{\psi x + 1}{\psi x - 1}$, then it becomes

$$\psi_1 x - \psi_1\left(\frac{x}{x-1}\right) = \frac{(x-1)^2 - 1}{x-1},$$

which is a particular case of

$$\psi_1 x + (a + v\,\phi x)\,\psi_1\left(\frac{x}{x-1}\right) = \frac{(x-1)^2 - 1}{x-1}$$

with which it agrees, when $a = -1$ and $v = 0$.

Put $\dfrac{x}{x-1}$ for x, then

$$\psi_1 \frac{x}{x-1} + \left(a + v\,\phi\frac{x}{x-1}\right)\psi_1 x = -\frac{(x-1)^2 - 1}{x-1},$$

whence by elimination,

$$\psi_1 x = \frac{(x-1)^2-1}{x-1} \cdot \frac{1+a+v\phi x}{1-(a+v\phi x)\left(a+v\phi\frac{x}{x-1}\right)},$$

and when $a=-1$, and $v=0$, this becomes

$$\psi_1 x = \frac{x^2-2x}{x-1} \cdot \frac{\phi x}{\phi x + \phi\left(\frac{x}{x-1}\right)},$$

and restoring the value of $\psi_1 x$, we find

$$\psi x = \frac{\left\{\frac{x^2-2x}{x-1} \cdot \frac{\phi x}{\phi x + \phi\left(\frac{x}{x-1}\right)}\right\}^{\frac{1}{n}} + 1}{\left\{\frac{x^2-2x}{x-1} \cdot \frac{\phi x}{\phi x + \phi\left(\frac{x}{x-1}\right)}\right\}^{\frac{1}{n}} - 1}$$

(22). $\psi x + \frac{x}{\sqrt{(1+x^2)}} \quad \psi\sqrt{(1-x^2)} = x;$

$$\psi x = \frac{x(1-x^2)\phi x}{x^2\phi\sqrt{(1-x^2)} + (1-x^2)\phi x}:$$

this solution was found by pursuing the course so frequently pointed out: another but not a more general one may be obtained as follows: multiply by $\sqrt{(1-x^2)}$; then

$$\sqrt{(1-x^2)}\,\psi x + x\,\psi\sqrt{(1-x^2)} = x\sqrt{(1-x^2)},$$

putting $\sqrt{(1-x^2)}\,\psi x = \psi_1 x$, we have

$$\psi_1 x + \psi_1\sqrt{(1-x)} = x\sqrt{(1-x^2)},$$

whose solution is $\psi_1 x = \frac{x\sqrt{(1-x^2)}}{2} + \phi x - \phi\sqrt{(1-x^2)}$,

hence

$$\psi x = \frac{x}{2} + \frac{\phi x - \phi\sqrt{(1-x^2)}}{\sqrt{(1-x^2)}}.$$

It would not be difficult to shew the identity of these two apparently different solutions.

(23). Given the equation

$$\frac{\psi x}{\sqrt{[\psi(1-x)]}} + \frac{\psi(1-x)}{\sqrt{(\psi x)}} = 1,$$

put $\psi_1 x = \dfrac{\psi x}{\sqrt{[\psi(1-x)]}}$, then $\psi_1(1-x) = \dfrac{\psi(1-x)}{\sqrt{(\psi x)}}$;

and the equation becomes

$$\psi_1 x + \psi_1(1-x) = 1,$$

whose general solution is

$$\psi_1 x = \frac{\phi x}{\phi x + \phi(1-x)};$$

hence

$$\frac{(\psi x)^2}{\psi(1-x)} = \frac{(\phi x)^2}{[\phi x + \phi(1-x)]^2}$$

putting $1-x$ for x, and eliminating $\psi(1-x)$, we find

$$\psi x = \frac{\{(\phi x)^2 \phi(1-x)\}^{\frac{2}{3}}}{[\phi(1-x) + \phi x]^2}.$$

(24). Given $(\psi x)^2 + \left(\psi \dfrac{a^2}{x}\right)^2 = \dfrac{x^4 + a^4}{x^2} \psi x . \psi \dfrac{a^2}{x}$;

divide by $\psi x . \psi \dfrac{a^2}{x}$ then

$$\frac{\psi x}{\psi \frac{a^2}{x}} + \frac{\psi \frac{a^2}{x}}{\psi x} = \frac{x^4 + a^4}{x^2};$$

putting $\psi_1 x = \dfrac{x}{\psi \frac{a^2}{x}}$, this becomes

$$\psi_1 x + \psi_1 \frac{a^2}{x} = \frac{x^4 + a^4}{x^2},$$

a particular solution of which is $\psi_1 x = x^2$; hence

$$\frac{\psi x}{\psi \frac{a^2}{x}} = x^2 \text{ or } \psi x = x^2 \psi \frac{a^2}{x}$$

and the general solution of this is

$$\psi x = x \chi \left(\overline{x}, \overline{\frac{a^2}{x}}\right).$$

(25). Given $\dfrac{\psi x + x}{\psi \frac{1}{x}} + x \dfrac{1 + x \psi \frac{1}{x}}{\psi x} = 1 + x^2,$

put $\psi_1 x = \dfrac{\psi x + x}{\psi \frac{1}{x}}$, then the equation becomes

$$\psi_1 x + x^2 \psi_1 \frac{1}{x} = 1 + x^2,$$

whose solution is $\psi_1 x = \dfrac{(x^2 + 1) \phi x}{\phi x + x^4 \phi \frac{1}{x}}$;

hence $\dfrac{x + \psi x}{\psi \frac{1}{x}} = \dfrac{(x^2 + 1) \phi x}{\phi x + x^4 \phi \frac{1}{x}}$;

putting $\frac{1}{x}$ for x and eliminating $\psi \frac{1}{x}$, we find

$$\psi x = \frac{(2x + x^{-1}) \phi x + x^5 \phi \frac{1}{x}}{(x^2 + x^{-2}) \phi x \,.\, \phi \frac{1}{x} - x^{-4} (\phi x)^2 - x^{-4} \left(\phi \frac{1}{x}\right)^2} \left(\phi \frac{1}{x} + x^{-4} \phi x\right)$$

(26). Given $(\psi x)^m . (\psi - x)^n - (\psi x)^n . (\psi - x)^m = 2x$; putting $\psi_1 x = (\psi x)^m . (\psi - x)^n$, it becomes

$$\psi_1 x - \psi_1(-x) = 2x,$$

whose solution is $\psi_1 x = x + \chi(\bar{x}, \overline{-x})$,

hence

$$(\psi x)^m . (\psi - x)^n . = x + \chi(\bar{x}, \overline{-x}),$$

and by the process for eliminating $\psi(-x)$, we shall find

$$\psi x = \left\{ \frac{\chi(\bar{x}, \overline{-x}) + x}{\{\chi(\bar{x}, \overline{-x}) - x\}^{\frac{m}{n}}} \right\}^{\frac{m}{m^2 - n^2}}.$$

(27). Given $\psi x + fx . \psi \alpha x = f_1 x$, where αx is such a function of x that $\alpha^2 x = x$; putting αx for x, we have

$$\psi \alpha x + f \alpha x . \psi x = f_1 \alpha x,$$

and by eliminating $\psi \alpha x$

$$\psi x = \frac{f_1 x - fx . f_1 \alpha x}{1 - fx . f \alpha x}.$$

If $fx . f\alpha x = 1$ and $f_1 x - fx . f_1 \alpha x = 0$, then the solution becomes a vanishing fraction; also the general value of $f_1 x$ is in that case $f_1 x = \sqrt{(fx)} . f_2(\bar{x}, \overline{\alpha x})$ and the equation becomes

$$\psi x + fx . \psi \alpha x = \sqrt{fx} . f_2(\bar{x}, \overline{\alpha x});$$

dividing this by $\sqrt{(fx)}$ and putting instead of $\frac{1}{\sqrt{fx}}$ its value $\sqrt{(f\alpha x)}$ derived from the equation $fx . f\alpha x = 1$, we have

$$\sqrt{(f\alpha x)} . \psi x + \sqrt{(fx)} . \psi \alpha x = f_2(\bar{x}, \overline{\alpha x}),$$

which is a symmetrical equation, whose general solution is

$$\sqrt{(f\alpha x)} . \psi x = \frac{f_2(\bar{x}, \overline{\alpha x}) . \phi x}{\phi x + \phi \alpha x},$$

hence

$$\psi x = \frac{\sqrt{(f x)} \cdot f_2(\overline{x, \; a x}) \cdot \phi x}{\phi x + \phi a x}.$$

(28). Given $a + b \psi x = \psi (a + b x)$,

$$\psi x = a \frac{1 - b^n}{1 - b} + b^n x.$$

(29). Given $\dfrac{a \psi x}{b + c \psi x} = \psi \left(\dfrac{a x}{b + c x}\right)$;

$$\psi x = \frac{a^n x}{b^n + c \dfrac{a^n - b^n}{a - b} x}.$$

(30). Given $\dfrac{\psi x}{\psi x - 1} = \psi \left(\dfrac{x}{x - 1}\right)$;

$$\psi x = \frac{x}{\dfrac{1 - (-1)^n}{2} x + (-1)^n},$$

where n is arbitrary; it may therefore be changed into any symmetrical function of x and $\dfrac{x}{x - 1}$.

(31). The three last examples are particular cases of the equation

$$a \psi x = \psi a x,$$

whose general solution is $\psi x = a^n x$.

(32). Given $\dfrac{\psi \left(\dfrac{x}{1 + x}\right)}{1 + \psi \left(\dfrac{x}{1 + x}\right)} = \psi \left(\dfrac{x}{1 + 2x}\right)$,

$$\psi x = \frac{x}{1 + n x}.$$

(33). Given $a\psi a x = \psi a^2 x$

$$\psi x = a^n x.$$

(34). Given $\psi x + a\psi\left(\frac{1}{1-x}\right) = \frac{1}{x}$;

$\frac{1}{1-x}$ is a periodic function of the third order, or of the form $a^3 x = x$; putting $\frac{1}{1-x}$ for x, we have

$$\psi\left(\frac{1}{1-x}\right) + a\psi\left(\frac{x-1}{x}\right) = 1-x,$$

and in this again putting $\frac{1}{1-x}$ for x, we find

$$\psi\left(\frac{x-1}{x}\right) + a\psi x = \frac{x}{x-1};$$

$\psi\left(\frac{1}{x-1}\right)$ and $\psi\left(\frac{x-1}{x}\right)$ being eliminated between these three equations, we have

$$\psi x = \frac{1}{1+a^3}\left\{\frac{1}{x} - a(1-x) + a^2\frac{x}{x-1}\right\}.$$

(35). Given $\psi x - a\psi\left(\frac{\sqrt{(x^2-1)}}{x}\right) = x^{2n}$,

put $\frac{\sqrt{(x^2-1)}}{x}$ for x, it becomes

$$\psi\left(\frac{\sqrt{(x^2-1)}}{x}\right) - a\psi\left(\frac{1}{\sqrt{(1-x^2)}}\right) = \frac{(x^2-1)^n}{x^{2n}}.$$

Again, put $\frac{\sqrt{(x^2-1)}}{x}$ for x, and we have

$$\psi\left(\frac{1}{\sqrt{(1-x^2)}}\right) - a\psi x = \frac{1}{(1-x^2)^n},$$

by eliminating $\psi\left(\frac{\sqrt{(x^2-1)}}{x}\right)$ and $\psi\left(\frac{1}{\sqrt{(1-x^2)}}\right)$ from these three equations, we shall find

$$\psi x = \frac{1}{1-a^3}\left\{x^{2n} + a\frac{(x^2-1)^n}{x^{2n}} + a^2\frac{1}{(1-x^2)^n}\right\}.$$

(36). Given $\psi x + \psi\left(\frac{1+x}{1-3x}\right) + \psi\left(\frac{x-1}{3x+1}\right) = a$

the function $\frac{1+x}{1-3x}$ is periodic of the third order, and by the process of elimination

$$\psi x = \frac{a\phi x + \phi_1 x - \phi_1\left(\frac{1+x}{1-3x}\right)}{\phi x + \phi\left(\frac{1+x}{1-3x}\right) + \phi\left(\frac{x-1}{3x+1}\right)}.$$

(37). Given $\frac{1}{\sqrt{\psi x}} + \frac{1}{\sqrt{\psi\frac{x-1}{x}}} + \frac{1}{\sqrt{\psi\frac{1}{1-x}}} = a.$

Putting $\psi_1 x = \frac{1}{\sqrt{\psi x}}$, we have

$$\psi_1 x + \psi_1\frac{x-1}{x} + \psi_1\frac{1}{1-x} = a,$$

whose solution is

$$\psi_1 x = \frac{a\phi x + \phi_1 x - \phi_1\frac{1}{1-x}}{\phi x + \phi\frac{1}{1-x} + \phi\frac{x-1}{x}},$$

hence

$$\psi x = \left\{\frac{\phi x + \phi\frac{1}{1-x} + \phi\frac{x-1}{x}}{a\phi x + \phi_1 x - \phi_1\frac{1}{1-x}}\right\}^2.$$

(38). Given $\psi x \, . \, \psi \frac{1}{1-x} \, . \, \psi \frac{x-1}{x} = c^3.$

Putting $\psi_1 x = \log \psi x$, we find

$$\psi_1 x + \psi_1 \frac{1}{1-x} + \psi_1 \frac{x-1}{x} = \log (c^3),$$

whose solution is found in the last problem. Changing ϕ_1 into $\log \phi_1$, we have

$$\psi x = \frac{\phi_1 x}{\phi_1 \frac{1}{1-x}} \log^{-1} \left(\frac{3 \, \phi x \, . \log c}{\phi x + \phi \frac{x-1}{x} + \phi \frac{1}{1-x}} \right).$$

Similarly if αx be any periodic equation of the third order.

(39). $\psi x + \psi \alpha x + \psi \alpha^2 x = a,$

has for its solution

$$\psi x = \frac{a \, \phi x + \phi_1 x - \phi_1 \alpha x}{\phi x + \phi \alpha x + \phi \alpha^2 x}.$$

(40). $\psi x \, . \, \psi \alpha x \, . \, \psi \alpha^2 x = c^3,$

has for its solution

$$\psi x = \frac{\phi_1 x}{\phi \alpha x} \log^{-1} \left\{ \frac{3 \, \phi x \, . \log c}{\phi x + \phi \alpha x + \phi \alpha^2 x} \right\}.$$

(41). Given $\psi x + f x \, . \, \psi \alpha x = f_1 x$, where $\alpha^3 x = x$,
Putting successively αx and $\alpha^2 x$ for x, we have

$$\psi \alpha x + f \alpha x \, . \, \psi \alpha^2 x = f_1 \alpha x,$$

$$\psi \alpha^2 x + f \alpha^2 x \, \psi x = f_1 \alpha^2 x,$$

and eliminating $\psi \alpha x$ and $\psi \alpha^2 x$ from these three equations, we have

$$\psi x = \frac{f_1 x - f x \, . f_1 \alpha x + f x \, . f \alpha x \, . f_1 \alpha^2 x}{1 + f x \, . f \alpha x \, . f \alpha^2 x}.$$

(42). Given $\psi x + fx . \psi \alpha x = f_1 x$ where $\alpha^n x = x$, a similar process of elimination will produce

$$\psi x = \frac{f_1 x - fx . f_1 \alpha x + \dots fx . f\alpha x . f\alpha^{n-2} x . f_1 \alpha^{n-1} x}{1 - (-1)^n fx . f\alpha x . . f\alpha^{n-1} x}.$$

(43). Given the equation

$$1 + fx . (\psi x + \psi \alpha x) - \psi x . \psi \alpha x = 0,$$

where αx is a periodic function of the second order, and fx is any function symmetrical relative to x and αx

$$fx = \frac{\psi x . \psi \alpha x - 1}{\psi x + \psi \alpha x},$$

consider ψx and $\psi \alpha x$ as two variables, and differentiate with respect to them, then

$$\frac{d \psi x}{1 + (\psi x)^2} + \frac{d \psi \alpha x}{(1 + \psi \alpha x)^2} = 0,$$

and by integration,

$$\tan^{-1} \psi x + \tan^{-1} \psi \alpha x = C = -\frac{1}{fx},$$

whose complete solution is

$$\tan^{-1} \psi x = \frac{\phi x}{\phi x + \phi \alpha x} \tan^{-1} \frac{-1}{fx},$$

hence

$$\psi x = \tan \left\{ \frac{\phi x}{\phi x + \phi \alpha x} \tan^{-1} \frac{-1}{fx} \right\},$$

this process is analogous to one employed by M. Laplace, for the integration of a similar equation of differences.

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* *Journal de l'Ecole Polytecnique*, Cah. 15.

(44). Given

$$\psi x + \psi\left(\frac{1+x}{1-x}\right) + \psi\left(-\frac{1}{x}\right) + \psi\left(\frac{x-1}{x+1}\right) = 1.$$

$\frac{1+x}{1-x}$ being a periodic function of the 4[th] order.

$$\psi x = \frac{\phi x}{\phi x + \phi\left(\frac{1+x}{1-x}\right) + \phi\left(\frac{-1}{x}\right) + \phi\left(\frac{x-1}{x+1}\right)}$$

$$+ \frac{x^2+1}{x-1}\chi\left(\bar{x}, \overline{\frac{1+x}{1-x}}, \overline{\frac{-1}{x}}, \overline{\frac{x-1}{x+1}}\right) +$$

$$+ \frac{x^2+1}{x.(1-x)}\chi\left\{\bar{x}, \overline{\frac{1+x}{1-x}}, \overline{\frac{-1}{x}}, \overline{\frac{x-1}{x+1}}\right\}.$$

(45). Given $\psi(x, y) + \psi\left(\frac{a^2}{x}, \frac{a^2}{y}\right) = 1,$

$$\psi(x, y) = \frac{\phi(x, y)}{\phi(x, y) + \phi\left(\frac{a^2}{x}, \frac{a^2}{y}\right)}.$$

(46). $\psi(x, y) + \psi\left(\frac{a^2}{x}, -y\right) = y^2,$

$$\psi(x, y) = \frac{y^2\phi(x, y)}{\phi(x, y) + \phi\left(\frac{a^2}{x}, -y\right)}.$$

(47). Given $\psi(x, y) + x^2\psi\left(\frac{a^2}{x}, \frac{y}{y-1}\right) = \frac{xy^2}{y-1},$

$$\psi(x, y) = \frac{(y-1)^{-1}xy^2\phi(x, y)}{\phi(x, y) + \phi\left(x, \frac{y}{y-1}\right)}.$$

(48). Given $\psi(x, y) + f(x, y)\psi(\alpha x, \beta y) =$

$$\sqrt{f(x, y)}.f_1(\bar{x}, \overline{\alpha x}, \bar{\overset{1}{y}}, \overline{\overset{1}{\beta y}}),$$

where $\alpha^2 x = x$, $\beta^2 y = y$ and $f(x, y)$ is such a function that

$$f(x, y).f(\alpha x, \beta y) = 1,$$

then $\psi(x, y) = \dfrac{\sqrt{f(\alpha x, \beta y)}.f_1(\bar{x}, \overline{\alpha x}, \bar{y}^1, \overline{\beta y}^1)}{f(x, y)\phi(\alpha x, \beta y) + f(\alpha x, \beta y)\phi(x, y)}$,

(49). Given $\psi(x, y) = \psi\left(\dfrac{x-y}{y}, \dfrac{x-y}{x}\right)$

$$\psi(x, y) = \phi\left(\frac{x}{y}\right).$$

(50). Given $\psi(x, y) = \psi\left(\dfrac{f(x, y)}{y}, \dfrac{f(x, y)}{x}\right)$;

$$\psi(x, y) = \phi\left(\frac{x}{y}\right).$$

(51). Given $\psi(x, y) = \psi\left(\dfrac{y}{2}\sqrt{\dfrac{y}{2x}}, \sqrt{2xy}\right)$,

$$\psi(x, y) = \chi\left(2xy + y^2, \frac{1}{y\sqrt{2xy}}\right).$$

(52). Given $\psi(x, y) = \psi(y, x)$

$$\psi(x, y) = \chi(x + y, xy).$$

(53). Given $\psi(x, y) = \left(\dfrac{x}{y}\right)^2 \psi(y, x)$

$$\psi(x, y) = \frac{x}{y^2}.\phi(\bar{x}, \bar{y}) = \frac{x}{y^2}.\phi_1(x + y, xy).$$

(54). Given $\psi(\pi - x) = \dfrac{d\psi x}{dx}$,

differentiating $\dfrac{d}{dx}\psi(\pi - x) = \dfrac{d^2\psi x}{dx^2}$, putting $\pi - x$ for x in the given equation

$$\psi(x) = -\frac{d\psi(\pi - x)}{dx},$$

and eliminating $\frac{d\psi(\pi - x)}{dx}$, we have

$$-\psi x = \frac{d^2 \psi x}{dx^2},$$

whence by integration

$$\psi x = b \cos x + c \sin x,$$

and it will be found that $c = -b$; hence

$$\psi x = b(\cos x - \sin x).$$

(55). Given $\psi(x, y) = \frac{d\psi(x, a-y)}{dx}$,

put $a - y$ for y, then

$$\psi(x, a-y) = \frac{d\psi(x, y)}{dx},$$

differentiate this relative to x, then

$$\frac{d\psi(x, a-y)}{dx} = \frac{d^2\psi(x, y)}{dx^2},$$

which being substituted in the given equation produces

$$\psi(x, y) = \frac{d^2\psi(x, y)}{dx^2},$$

whose solution is

$$\psi(x, y) = \epsilon^x \phi y + \epsilon^{-x} \phi_1 y,$$

ϕ and ϕ_1 being two arbitrary functions so constituted as to fulfil the given equation, in order to determine them, put $a-y$ for y and differentiate relative to x, then

$$\frac{d\psi(x, a-y)}{dx} = \epsilon^x \phi(a-y) - \epsilon^{-x} \phi_1(a-y)$$

hence

$$\phi y = \phi(a - y) \text{ and } \phi_1 y = -\phi_1(a - y),$$

whose solutions are

$$\phi x = \chi(\bar{y}, \overline{a - y}) \text{ and } \phi_1 y = (a - 2y)\chi_1(\bar{y}, \overline{a - y}),$$

hence the general solution of the equation is

$$\psi(x, y) = \epsilon^x \chi(\bar{y}, \overline{a-y}) + \epsilon^{-x}(a - 2y)\chi_1(\bar{y}, \overline{a - y})$$

A similar mode of solution is applicable to the three following equations.

(56). Given $\psi(x, y) = \frac{d}{dx}\psi\left(x, \frac{1}{y}\right)$

$$\psi(x, y) = \epsilon^x \phi\left(\bar{y}, \overline{\frac{1}{y}}\right) + \epsilon^{-x}\frac{1-y^2}{y}\phi_1\left(\bar{y}, \overline{\frac{1}{y}}\right).$$

(57). Given $\psi(x, y) = \frac{d}{dx}\psi\left(x, \frac{y}{y-1}\right)$

$$\psi(x, y) = \epsilon^x \phi\left(\bar{y}, \overline{\frac{y}{y-1}}\right) + \epsilon^{-x}\frac{2y - y^2}{y-1}\phi_1\left(\bar{y}, \overline{\frac{y}{y-1}}\right).$$

(58). Given $\psi(x, y) = \frac{d}{dx}\psi(x, \alpha y)$, where $\alpha^2 y = y$.

$$\psi(x, y) = \epsilon^x \phi(y, \alpha y) + \epsilon^{-x}(\alpha y - y)\phi_1(\bar{y}, \overline{\alpha y}).$$

(59). Given $\psi(x, y) = \frac{d\psi(x, \alpha y)}{dx}$

where α is such a function that $\alpha^4 y = y$.

Substituting successively αy, $\alpha^2 y$, $\alpha^3 y$ for y, we have

$$\psi(x, \alpha y) = \frac{d\psi(x, \alpha^2 y)}{dx},$$

$$\psi(x,\ a^2 y) = \frac{d\,\psi(x,\ a^3 y)}{d\,x},$$

$$\psi(x,\ a^3 y) = \frac{d\,\psi(x,\ y)}{d\,x}.$$

From the given equation $\frac{d\,\psi(x,\ a\,y)}{d\,x}$ may be eliminated by means of the second, and from the result $\frac{d\,\psi(x,\ a^2 y)}{d\,x}$ may be eliminated by the third equation, and continuing this, we should find

$$\psi(x,\ y) = \frac{d^4\,\psi(x,\ y)}{d\,x^4},$$

the solution of this equation is

$$\psi(x,\ y) = \epsilon^x\,\phi\,y + \epsilon^{-x}\,\phi_1\,y + \sin x\,.\,\phi_2 y + \cos x\,.\,\phi_3\,y,$$

ϕ, ϕ_1, ϕ_2, ϕ_3 must be determined so as to satisfy the given equation, taking the differential and putting $a\,y$ for y, we have

$$\frac{d\,\psi(x,\ a\,y)}{d\,x} = \epsilon^x\,\phi\,a\,y - \epsilon^{-x}\,\phi_1\,a\,y + \cos x\,.\,\phi_2\,a\,y - \sin x\,.\,\phi_3\,a\,y$$

the first condition to satisfy is

$$\phi\,y = \phi\,a\,y,$$

which gives

$$\phi\,y = \chi(\overline{y},\ \overline{a\,y},\ \overline{a^2 y},\ \overline{a^3 y}),$$

the next condition is

$$\phi_1 y = -\ \phi_1\,a\,y$$

whose solution is

$$\phi_1 y = (a^3 y - a^2 y + a\,y - y\,(\chi_1\,(\overline{y},\ \overline{a\,y},\ \overline{a^2 y},\ \overline{a^3 y}),$$

the other two conditions are

$$\phi_2 y = -\ \phi_3\,a\,y,\ \text{and}\ \phi_3\,y = \phi_2\,a\,y$$

putting αy for y in the second of these it becomes $\phi_2 \alpha y = \phi_2 \alpha^2 y$ and this substituted in the first gives,

$$\phi_2 y = - \phi_2 \alpha^2 y,$$

whose solution is $\phi_2 y = (\alpha^2 y - y) \chi_2 (\overline{y}, \overline{\alpha y}^{1}, \overline{\alpha^2 y}, \overline{\alpha^3 y}^{1})$

hence

$$\phi_3 y = (\alpha^3 y - \alpha y) \chi_2 (\overline{\alpha y}, \overline{\alpha^2 y}^{1}, \overline{\alpha^3 y}, \overline{y}^{1}),$$

and the general solution of the equation is

$$\psi(x, y) = \epsilon^x \chi (\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}) +$$

$$+ \epsilon^{-x} (\alpha^3 y - \alpha^2 y + \alpha y - y) \chi_1 (\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}) +$$

$$+ (\alpha^2 y - y) \chi_2 (\overline{y}, \overline{\alpha y}^{1}, \overline{\alpha^2 y}, \overline{\alpha^3 y}^{1}) \sin x +$$

$$+ (\alpha^3 y - \alpha y) \chi_2 (\overline{\alpha y}, \overline{\alpha^2 y}^{1}, \overline{\alpha^3 y}, \overline{y}^{1}) \cos x.$$

(60). Given the equation $\psi(x, y) = \frac{d^n \psi(x, \alpha y)}{dx^n}$,

where α is such a function that $\alpha^n x = x$.

This equation may be reduced to the solution of the partial differential equation

$$\psi(x, y) = \frac{d^{pn} \psi(x, y)}{dx^{pn}},$$

and the arbitrary functions of y which occur in its solution, must be determined by the conditions of the equation.

(61). Given the equation

$$\frac{d \psi(a - x, y)}{dy} = \frac{d \psi(x, b - y)}{dx},$$

put $a - x$ for x, also $b - y$ for y, then we have the two equations

$$\frac{d\psi(x,\, y)}{dy} = -\frac{d\psi(a-x,\, b-y)}{dx},$$

$$-\frac{d\psi(a-x,\, b-y)}{dy} = \frac{d\psi(x,\, y)}{dx}.$$

If the first of these be differentiated relative to y, and the second relative to x; then the right side of the first resulting equation will be identical with the left side of the second, and we shall have

$$\frac{d^2\psi(x,\, y)}{dy^2} = \frac{d^2\psi(x,\, y)}{dx^2};$$

the solution of this partial differential equation is

$$\psi(x,\, y) = \phi(x+y) + \phi_1(x-y);$$

the two arbitrary functions must be determined so as to satisfy the equation; we have

$$\frac{d\psi(a-x,\, y)}{dy} = \phi'(a-x+y) - \phi'_1(a-x-y),$$

$$\frac{d\psi(x,\, b-y)}{dx} = \phi'(b+x-y) + \phi'_1(-b+x+y),$$

ϕ' and ϕ'_1 being the differential coefficients of ϕ and ϕ_1 these two expressions must be identical, hence

$$\phi'(a-\overline{x-y}) = \phi'(b+\overline{x-y}),$$

and

$$-\phi'_1(a-\overline{x+y}) = \phi'_1(-b+\overline{x+y}),$$

the solutions of which equations are

$$\phi'(x+y) = \chi\,\{\overline{x+y},\ \overline{a-b-x-y}\},$$

and

$$\phi'_1(x+y) = (a-b-2x-2y)\,\chi_1\,\{\overline{x+y},\ \overline{a-b-x-y}\}$$

and substituting these values, we have

$$\psi(x, y) = \int (dx + dy)\,\chi\,\{\overline{x+y},\ \overline{a-b-x-y}\} +$$
$$+ \int (dx - dy)(a-b-2x-2y)\,\chi_1\,\{\overline{x-y},\ \overline{a-b-2x+2y}\}\,.$$

(63). Given the equation

$$\frac{d\,\psi\left(x, \frac{1}{y}\right)}{dx} = \frac{d\,\psi\left(\frac{1}{x}, y\right)}{dy}.$$

Put $\frac{1}{y}$ for y and differentiate relative to x, then

$$\frac{d^2\,\psi(x, y)}{dx^2} = -\frac{d^2\,\psi\left(\frac{1}{x}, \frac{1}{y}\right)}{dx\,dy}\,y^2.$$

Again, put $\frac{1}{x}$ for x, and differentiate relative to y, then

$$-\frac{d^2\,\psi\left(\frac{1}{x}, \frac{1}{y}\right)}{dx\,dy}\,x^2 = \frac{d^2\,\psi(x, y)}{dy^2}$$

hence

$$\frac{d^2\,\psi(x, y)}{dy^2} = \frac{x^2}{y^2}\,\frac{d^2\,\psi(x, y)}{dx^2},$$

the solution of this equation of partial differentials is

$$\psi(x, y) = x\,\phi\left(\frac{x}{y}\right) + \phi_1(xy):$$

to determine the form of ϕ and ϕ_1, we have

$$\frac{d\,\psi\left(x, \frac{1}{y}\right)}{dx} = \phi(xy) + xy\,\phi'(xy) + \frac{1}{y}\,\phi'_1\left(\frac{x}{y}\right),$$

$$\frac{d\,\psi\left(\frac{1}{x}, y\right)}{dy} = -\frac{1}{x^2 y}\,\phi'\left(\frac{1}{xy}\right) + \frac{1}{x}\,\phi'_1\left(\frac{y}{x}\right).$$

In order that these two expressions may coincide, we must have

$$\phi(xy) + xy\,\phi'(xy) = -\frac{1}{x^2y^2}\,\phi'\left(\frac{1}{xy}\right)$$

$$\phi'_1\left(\frac{x}{y}\right) = \frac{y}{x}\,\phi'_1\left(\frac{y}{x}\right).$$

The first of these multiplied by $d(xy)$ may be put under the form

$$d(xy)\,.\,\phi(xy) + xy\,\frac{d\phi(xy)}{d(xy)}\,d(xy) = d\,\phi\left(\frac{1}{xy}\right)$$

whose integral is

$$xy\,\phi(xy) = \phi\left(\frac{1}{xy}\right),$$

the solution of which functional equation is

$$\phi(xy) = \frac{1}{\sqrt{xy}}\,\chi\left(\overline{xy},\ \overline{\frac{1}{xy}}\right)$$

the solution of the second equation is

$$\phi'_1\left(\frac{x}{y}\right) = \sqrt{\frac{y}{x}}\,.\,\chi_1\left(\overline{\frac{x}{y}},\ \overline{\frac{y}{x}}\right),$$

employing these values of ϕ and ϕ_1, we have

$$\psi(x,\ y) = \sqrt{xy}\,.\,\chi\left(\overline{\frac{x}{y}},\ \overline{\frac{y}{x}}\right) +$$

$$\int d(xy)\,.\left(\frac{1}{xy}\right)^{\frac{1}{2}}\chi_1\left(\overline{xy},\ \overline{\frac{1}{xy}}\right).$$

(63). Given $\dfrac{d\psi(x,\ \alpha y)}{dx} = \dfrac{d\psi(\beta x,\ y)}{dy}$,

where $\alpha^2 y = y$ and $\beta^2 x = x$ a process nearly similar to that by which the two last equations were solved will lead to the partial differential equation

$$\frac{d\alpha y}{dy} \cdot \frac{d^2 \psi(x, y)}{dx^2} = \frac{d\beta x}{dx} \cdot \frac{d^2 \psi(x, y)}{dy^2}.$$

(64). Given $\psi \alpha x = \psi \psi x = \psi^2 x.$

It is evident, that whatever be the form of α, this equation can always be satisfied by assuming $\psi x = \alpha x$, hence the solutions of the following equations,

$$\psi(-x) = \psi^2 x \qquad \psi x = -x$$

$$\psi\left(\frac{ax}{b+cx}\right) = \psi^2 x \qquad \psi x = \frac{ax}{b+cx}$$

$$\psi\sqrt{\frac{1+x}{x}} = \psi^2 x \qquad \psi x = \sqrt{\frac{1+x}{x}},$$

(65). Given $\psi(2a - x) = \psi^3 x.$

Put $\psi x = \phi^{-1} f \phi x,$

then $\psi^2 x = \phi^{-1} f \phi \phi^{-1} f \phi x = \phi^{-1} f^2 \phi x,$

and $\psi^3 x = \phi^{-1} f^2 \phi \phi^{-1} f \phi x = \phi^{-1} f^3 \phi x,$

and the equation becomes

$$\phi^{-1} f \phi(2a - x) = \phi^{-1} f^3 \phi x.$$

This equation may be satisfied in the following manner: by making f a periodic function of the second order, we have $f^2 v = v$, and the equation becomes

$$\phi^{-1} f \phi(2a - x) = \phi^{-1} f \phi x,$$

or

$$\phi(2a - x) = \phi x,$$

This is satisfied by making ϕ any symmetrical function of x and $2a - x$. As an example take $fv = -v$, also

$$\phi x = x . \overline{2a - x} = 2ax - x^2,$$

then

$$\phi^{-1} x = a \pm \sqrt{(a^2 - x)},$$

and

$$\psi x = \phi^{-1} f \phi x = a \pm \sqrt{a^2 - \frac{1}{x^2 - 2ax}},$$

(66). Given $\psi\left(\frac{1-x}{1+x}\right) = \psi^3 x$

$\psi x = \phi^{-1} f \phi x$ where ϕ and f are determined by the equations

$$\phi x = \chi \left\{ \overline{\frac{1-x}{1+x}}, \bar{x} \right\} \text{ and } f^2 x = x.$$

(67). Given $\psi a x = \psi^3 x$, where $a^n v = v$

putting $\psi x = \phi^{-1} f \phi x$, we have

$$\phi^{-1} f \phi a x = \phi^{-1} f^3 \phi x$$

determine ϕ from the condition $\phi x = \phi a x$; hence,

$$\phi x = \chi \{ \bar{x}, \overline{ax}, \ldots\ldots \overline{a^{n-1} x} \},$$

and let f be such a function that $f^2 x = x$, then the equation is satisfied.

(68). Given $\psi^p a x = \psi^q x$ where $q > p$ and $a^n x = x$, the substitution $\phi^{-1} f \phi x$ instead of ψ will give,

$$\phi^{-1} f^p \phi a x = \phi^{-1} f^q \phi x,$$

and this is satisfied if $\phi x = \chi (\bar{x}, \overline{ax}, \overline{a^{n-1} x})$

and also $f^{q-p} v = v$, for it then becomes

$$\phi^{-1} f^p \phi a x = \phi^{-1} f^p \phi x, \text{ where } \phi x = \phi a x.$$

If in a function of two variables, as $\psi(x, y)$, we substitute the function itself instead of one of those quantities, the result is denoted thus,

$$\psi\{x, \psi(x, y)\} = \psi^{1,2}(x, y),$$

$$\psi\{\psi(x, y), y\} = \psi^{2,1}(x, y),$$

if the function itself is substituted simultaneously for x and y, it is denoted thus

$$\psi\{\psi(x, y), \psi(x, y)\} = \psi^{\overline{2,2}}(x, y).$$

(69). Given $\psi^{\overline{2,2}}(x, y) = a$,

By means of the substitution $\phi^{-1} f \phi x$, for ψx, we are enabled to reduce functional equations of any order to those of the first, a substitution nearly resembling it, will be of equal value for those which contain two or more variables, by assuming

$$\psi(x, y) = \phi^{-1} f(\phi x, \phi y),$$

we have

$$\psi^{\overline{2,2}}(x, y) = \phi^{-1} f\{\phi\phi^{-1} f(\phi x, \phi y), \phi\phi^{-1} f(\phi x, \phi y)\}$$

$$= \phi^{-1} f^{\overline{2,2}}(\phi x, \phi y),$$

and substituting this value in the equation

$$\phi^{-1} f^{\overline{2,2}}(\phi x, \phi y) = a.$$

Put $\phi^{-1} x$ for x, and ϕ^{-1} for y, also taking the function ϕ on both sides

$$f^{\overline{2,2}}(x, y) = \phi a.$$

If therefore we are acquainted with a particular solution, we find the general one; let the function $A\frac{x}{y}$ be tried, then

$$f^{\overline{2,2}}(x,\ y) = A\,\frac{A^{\frac{x}{y}}}{A^{\frac{x}{y}}} = \phi\,a$$

hence $A = \phi\,a$, and the solution is

$$\psi\,(x,\ y) = \phi^{-1}\left(\frac{\phi\,x}{\phi\,y}\,\phi\,a\right),$$

a variety of solutions may be found of different forms, such as

$$\psi\,(x,\ y) = \frac{a}{\phi\,(1)}\,\phi\left(\frac{x}{y}\right),\quad \psi\,(x,\ y) = \frac{a}{\phi\,(1)}\,\phi\left(\frac{\alpha\,(x,\ y)}{\beta(x,\ y)}\right),$$

where α and β are any two homogeneous functions of the same degree.

(70). If $\psi\,(x,y) = a\,x + b\,y$,

then $\psi^{\overline{n,n}}\,(x,\ y) = (a+b)^{n-1}\,(ax+by)$,

(71). If $\psi\,(x,\ y)$ is any homogeneous function of x, and y of the degree n,

then

$$\psi^{\overline{k,k}}(x,\ y) = \{\,\psi\,(x,\ y)\,\}^{n^{k-1}} \times \{\,\psi\,(1,\ 1)\,\}^{\frac{1-n^{k-1}}{1-n}}.$$

(72). Given $\psi^{\overline{2,2}}(x,\ y) = \sqrt{\psi\,(x,\ y)}$,

$$\psi\,(x,\ y) = \sqrt{\frac{x^2+y^2}{2\,y\,\phi\left(\frac{x}{y}\right)}}\,\phi\,(1).$$

(73). Given

$$\psi^{\overline{2,2}}(x,\ y) = \psi\,(x,\ y) + \frac{1}{\psi\,(x,\ y)},$$

$$\psi(x, y) = \frac{2\phi\left(\frac{x}{y}\right)}{(x+y)\phi(1)} + \frac{(x+y)\phi(1)}{2\phi\left(\frac{x}{y}\right)}.$$

(74). Given $\psi^{\overline{2,2}}(x, y) = \frac{1 - \psi(x, y)}{1 + \psi(x, y)}$,

$$\psi(x, y) = \frac{y\phi(1) - x^2\phi\left(\frac{x}{y}\right)}{y\phi(1) + x^2\phi\left(\frac{x}{y}\right)}.$$

(75). $\psi^{\overline{2,2}}(x, y) = F\psi(x, y)$,

$$\psi(x, y) = F\left(\frac{\alpha(x, y)}{\beta(x, y)}\right),$$

provided α and β are homogeneous with respect to x and y; the first of the $n + 1$ degree, the second of the n^{th}, and also at the same time $\alpha(1, 1) = \beta(1, 1)$.

(76). Given $\psi^{\overline{2,2}}(x, y) = F\psi(x, y)$.

Another solution of the same equation is

$$\psi(x, y) = F\left(\frac{\alpha(x, y)}{\beta(x, y)}x\right),$$

where α and β are two such functions, that when $x = y$, we have also

$$\alpha(x, y) = \beta(x, y).$$

(77). Given $\psi^{3,3}(x, y) = \psi(x, y)$,

$$\psi(x, y) = \left\{a - \left(\frac{x^2 + y^2}{x + y}\right)^n\right\}^{\frac{1}{n}}.$$

(78). Given $\psi^{\overline{n,n}}(x,\ y) = \{\psi(x,\ y)\}^{m}$

$$\psi(x,\ y) = \left\{\frac{2xy\,\phi(1)}{(x+y)\,\phi\left(\frac{x}{y}\right)}\right\}^{m^{\frac{1}{n-1}}}.$$

(79.) Given $\psi^{\overline{4,4}}(x,\ y) = \{\psi^{\overline{2,2}}(x,\ y)\}^{2}$

$$\psi(x,\ y) = \left\{\frac{(x+y)\,\phi(1)}{2\,\phi\left(\frac{x}{y}\right)}\right\}^{\sqrt{2}}.$$

(80). Given $\psi^{\overline{3,3}}(x,\ y) = \frac{x^2}{y}$.

$$\psi(x,\ y) = \phi^{-1}\left\{\frac{1+\phi\frac{x^2}{y}}{1-3\,\phi\frac{x^2}{y}}\right\}.$$

(81). Given $x\,\psi^{1,2}(x,\ y) = y\,\psi^{2,1}(x,\ y)$,
put $\phi^{-1}f(\phi x,\ \phi y)$ for ψx, then it becomes

$$x\,\phi^{-1}f^{1,2}(\phi x,\ \phi y) = y\,\phi^{-1}f^{2,1}(\phi x,\ \phi y);$$

putting $\phi^{-1}x$ for x, and $\phi^{-1}y$ instead of y, we have

$$\phi^{-1}x\,.\,\phi^{-1}f^{1,2}(x,\ y) = \phi^{-1}y\,.\,\phi^{-1}f^{2,1}(x,\ y);$$

if $f^{1,2}(x,\ y) = y$, and $f^{2,1}(x,\ y) = x$, this equation, becomes identical; but making $f(x,y) = a - x - y$, these two equations are verified; consequently the general solution is

$$\psi(x,\ y) = \phi^{-1}(a - \phi x - \phi y).$$

(82). Given $\psi^{1,2}(x,\ y)\cdot\psi^{2,1}(x,\ y) = xy$,

$$\psi(x,\ y) = \phi^{-1}\left(\frac{a}{\phi x\,.\,\phi y}\right).$$

(83). Given $x\,\psi^{\overline{2,2}}(x,\ y) = a\,\psi^{2,1}(x,\ y)$,

$$\psi(x,\ y) = \phi^{-1}\left(\frac{\phi a\,.\,\phi y}{\phi x}\right).$$

Various methods for the solution of Functional Equations may be found in the following writings:

Speculationes Analytico Geometricæ, *N. Fuss.* Mem. de l'Acad. Imp. de St. Petersburg, Vol. IV. p. 225. 1811.

Memoirs of the Analytical Society, p. 96. 1813.

Observations on various points of Analysis, Phil. Trans. *J. F. W. Herschel.*

Essay towards the Calculus of Functions, *C. Babbage.* 1815.

Ditto, Part II. p. 179. 1816.

Observations on the analogy which subsists between the Calculus of Functions, and other branches of Analysis, Phil. Trans. 1817. p. 197. *C. Babbage.*

Spence's Essays, 1819. Note by *J. F. W. Herschel,* p. 151.

Annals of Philosophy, Nov. 1817. *Mr. Horner.*

Journal of the Royal Institution. *C. Babbage.*

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